Snakes, Chains, and Kernels

a primer on homological algebra

Jake Laubacher

St. Norbert College

Math Colloquium Series – February 14, 2019

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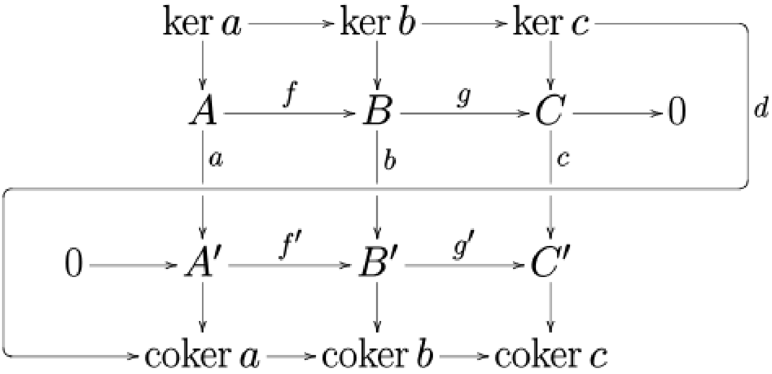
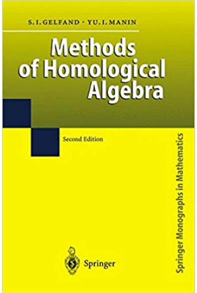
Snake Lemma:
`https://www.youtube.com/watch?v=BcWxqql4iOw`

The Snake Lemma



(Lack of) Proof



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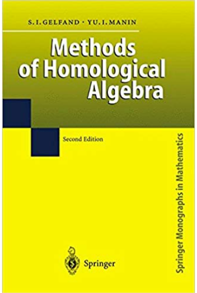

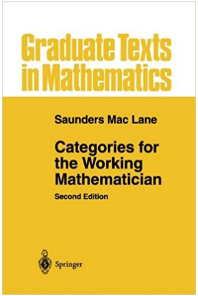

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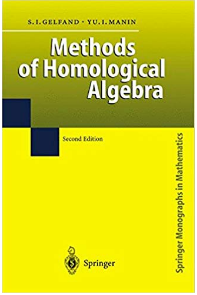

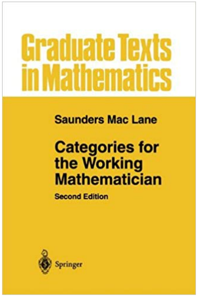

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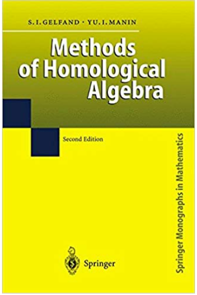

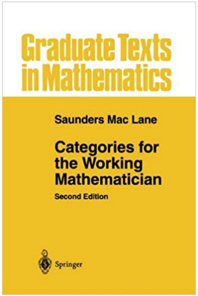

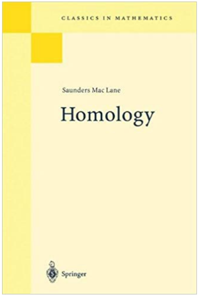

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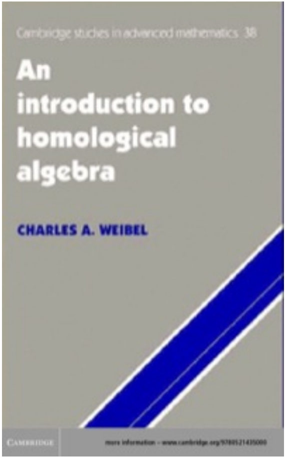

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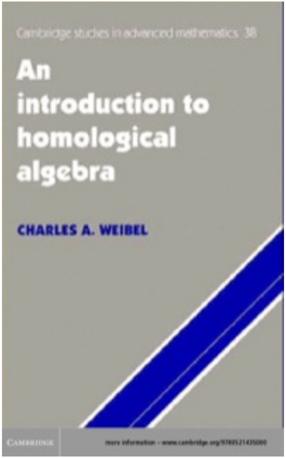

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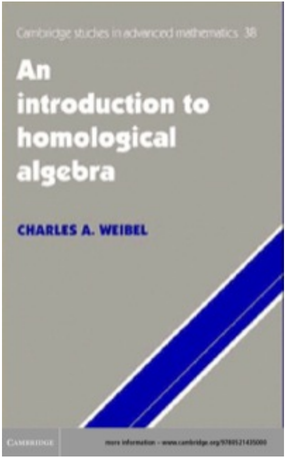

http://sites.math.rutgers.edu/~weibel/Hbook-corrections.html

A Famous Quote

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– Kayley Sjoholm, 2019

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Let R be a ring with multiplicative identity. (Think $\mathbb{Z}$.)

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(i) $m + n \in M$,

(ii) $m + n = n + m$,

(iii) $(m + n) + p = m + (n + p)$,

(iv) $m + 0_M = m$,

(v) $m + (-m) = 0_M$,

(vi) $r \cdot m \in M$,

(vii) $r \cdot (m + n) = r \cdot m + r \cdot n$,

(viii) $(r + s) \cdot m = r \cdot m + s \cdot m$,

(ix) $r \cdot (s \cdot m) = (rs) \cdot m$, and

(x) $1_R \cdot m = m$.

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Example

Vector spaces are modules over a field.

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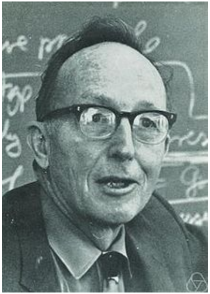

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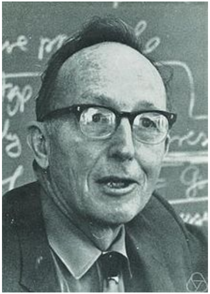

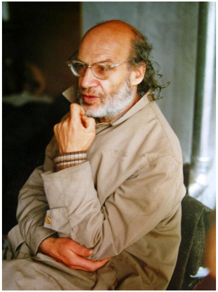

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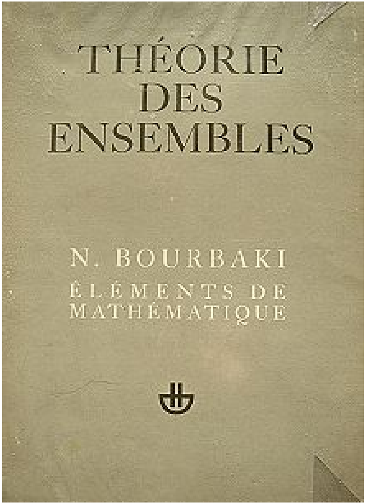

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The homomorphism $f : M \longrightarrow N$ is **one-to-one** if $\ker(f) = \{0\}$.

Proof.

Recall that $\ker(f) = \{m \in M \mid f(m) = 0\}$. Observe that $f(0) = f(0+0) = f(0) + f(0)$, and so $f(0) = 0$. Moreover, if $f(a) = f(b)$, then

$$f(a) - f(b) = f(a-b) = 0.$$

So f is one-to-one if $a = b$, in which case $a - b = 0$. □

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Definition

We say that the sequence of maps

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is **exact** at B if $\mathrm{Im}(f) = \ker(g)$.

The Five Lemma

Lemma

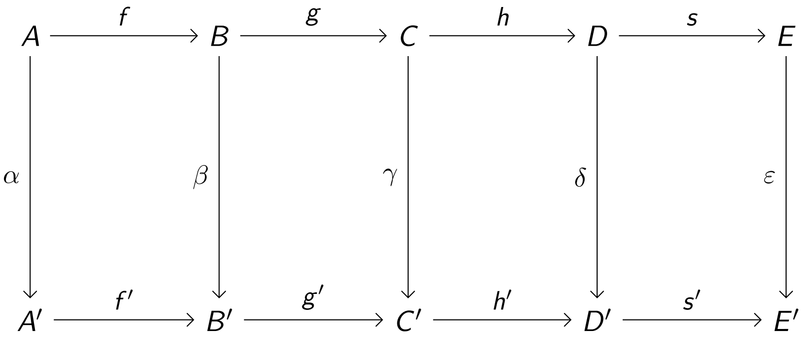
In any commutative diagram

$$\begin{array}{ccccccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D & \xrightarrow{s} & E \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\varepsilon} \\
A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & D' & \xrightarrow{s'} & E'
\end{array}$$

with exact rows, and isomorphisms α, β, δ, and ε, we must also have that γ is an isomorphism.

Diagram Chase

Take $c \in C$ such that $\gamma(c) = 0$.
To show that γ is one-to-one, we will show that $\ker(\gamma) = \{0\}$.
Thus, we are done if we show $c = 0$.



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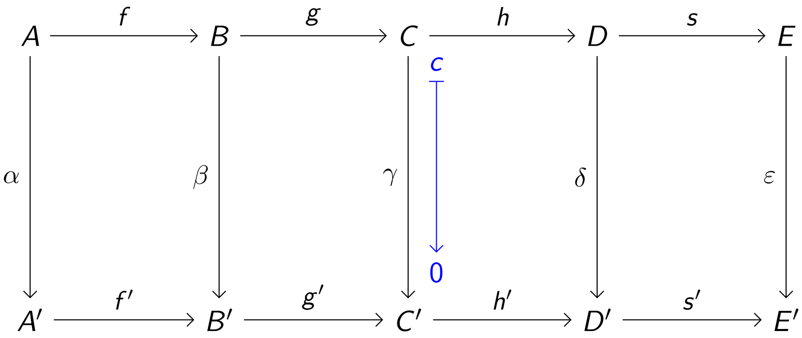

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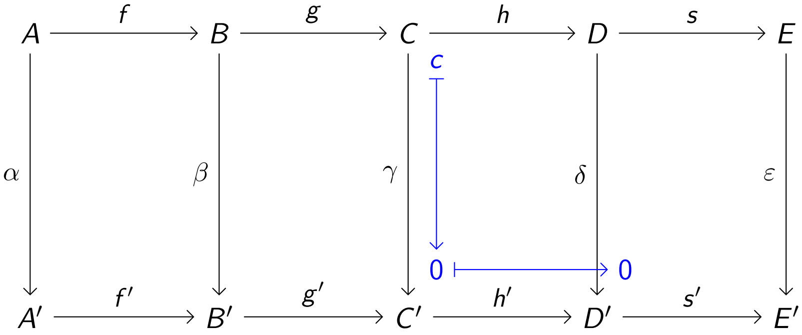

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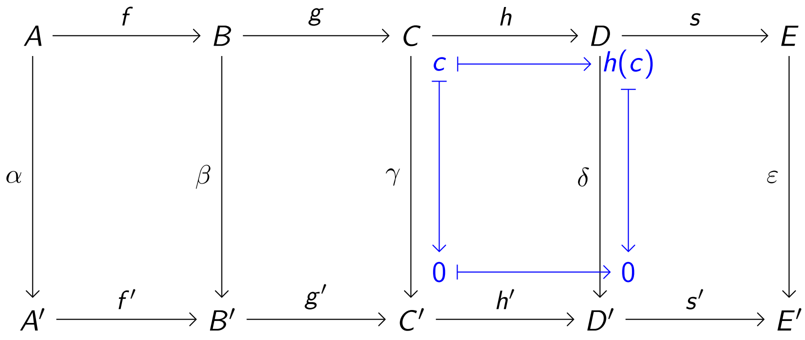

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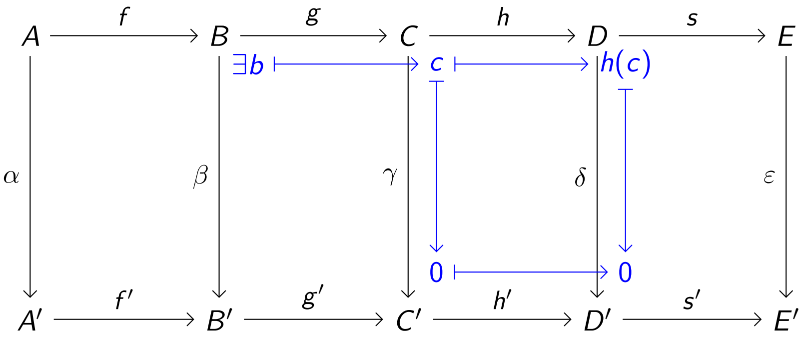

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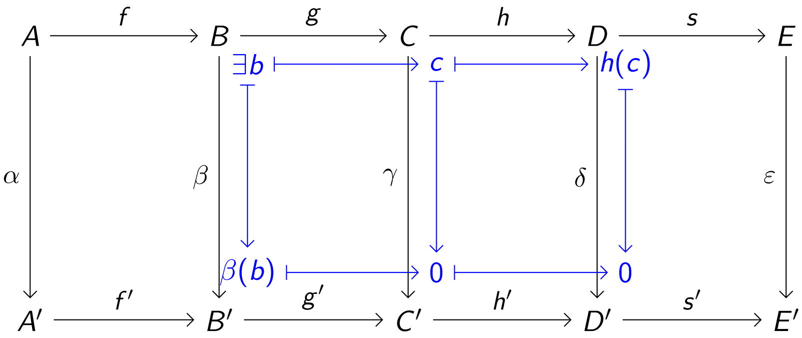

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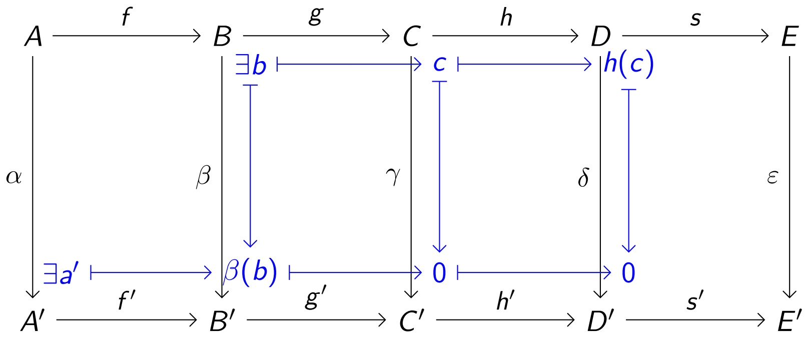

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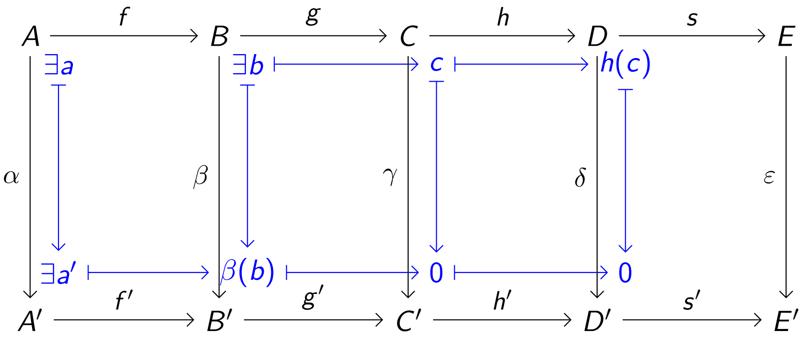

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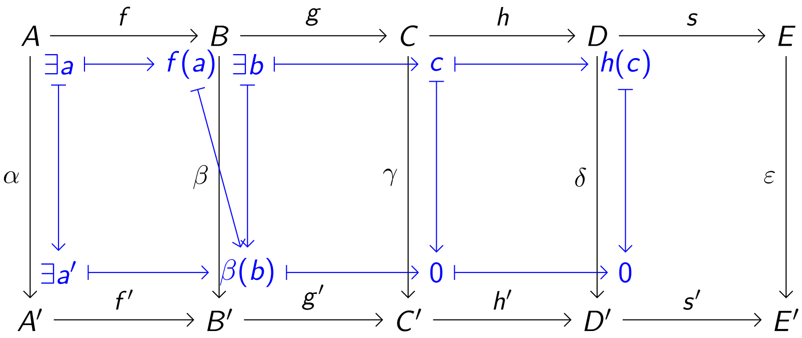

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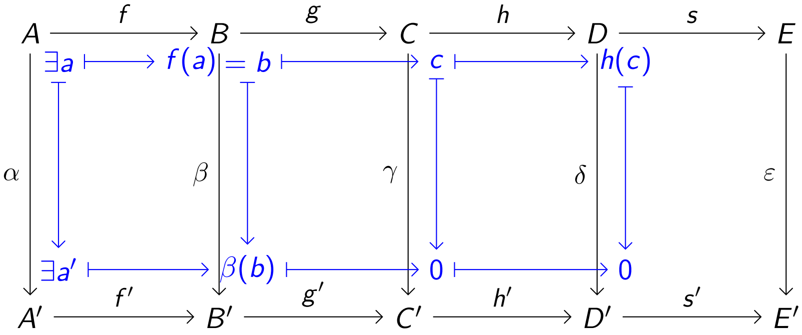

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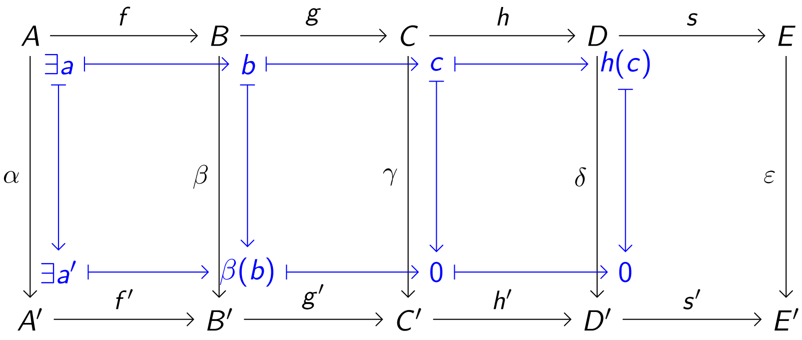

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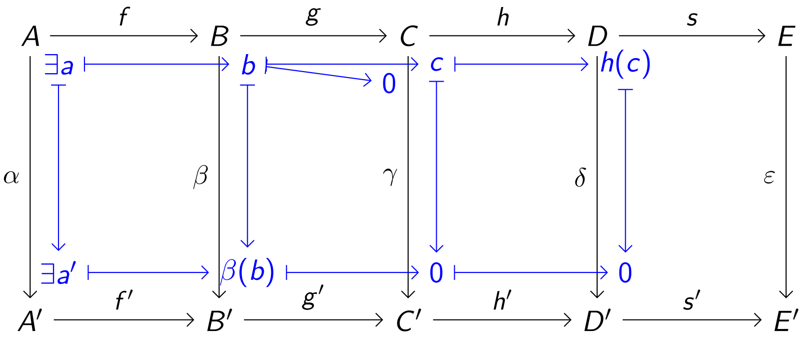

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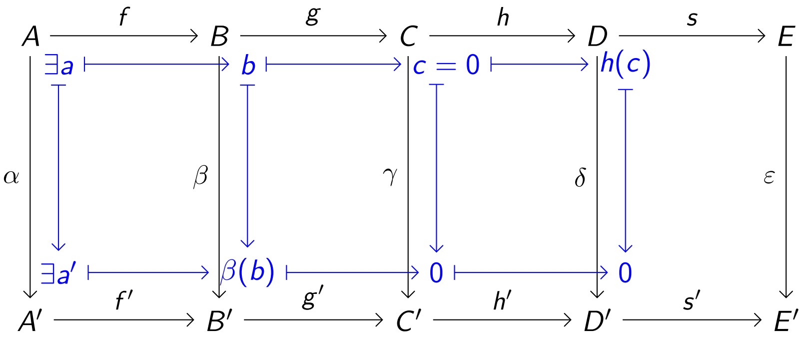

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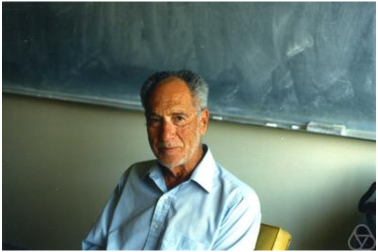

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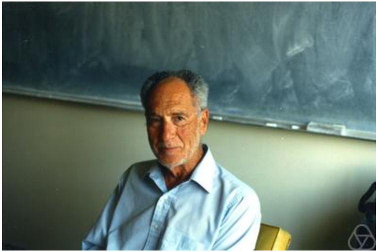

Questions?

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