A Deformation Theory Controlled by the higher order Hochschild cohomology $H^{\bullet}_{S^d}(A, A)$

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Higher order Hochschild cohomology

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Consider the chain

$$0 \longrightarrow A \xrightarrow{\delta_0} \dots \xrightarrow{\delta_{d-2}} A \xrightarrow{\delta_{d-1}}$$

$$\xrightarrow{\delta_{d-1}} \mathrm{Hom}(A,A) \xrightarrow{\delta_d} \mathrm{Hom}(A^{\otimes(d+1)},A) \xrightarrow{\delta_{d+1}} \dots$$

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Definition (Anderson – 1971, Pirashvili – 2000)

The cohomology of the chain complex above is called the **higher order Hochschild cohomology of A with coefficients in A** over the d-sphere. It is denoted by $\mathrm{H}^{\bullet}_{S^d}(A, A)$.

Higher order Hochschild cohomology

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Note that:

$$\begin{aligned}\delta_d(f)(a_1 \otimes \cdots \otimes a_{d+1}) &= a_1 \cdots a_d f(a_{d+1}) \\ &+ \sum_{i=1}^{d} (-1)^i a_1 a_2 \cdots \overline{a_{d+1-i} a_{d+2-i}} \cdots a_d a_{d+1} f(a_{d+1-i} a_{d+2-i}) \\ &+ (-1)^{d+1} a_2 \cdots a_{d+1} f(a_1).\end{aligned}$$

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Example $(d = 1)$

$$\delta_1(f)(a_1 \otimes a_2) = a_1 f(a_2) - f(a_1 a_2) + a_2 f(a_1)$$

Higher order Hochschild cohomology

Example ($d = 2$)

$$\delta_2(f)(a_1 \otimes a_2 \otimes a_3) = a_1 a_2 f(a_3) - a_1 f(a_2 a_3) + a_3 f(a_1 a_2) - a_2 a_3 f(a_1)$$

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Example ($d=3$)

$$\begin{aligned}\delta_3(f)(a_1 \otimes a_2 \otimes a_3 \otimes a_4) = {} & a_1 a_2 a_3 f(a_4) - a_1 a_2 f(a_3 a_4) \\ & + a_1 a_4 f(a_2 a_3) - a_3 a_4 f(a_1 a_2) \\ & + a_2 a_3 a_4 f(a_1)\end{aligned}$$

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Question (Corrigan-Salter – 2017)

What role does the higher order Hochschild cohomology play in deformation theory?

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Theorem (Carolus and Staic – 2018)

Let A be a commutative k-algebra and $u : A[[t]] \longrightarrow A[[t]]$ determined by $u(a) = a + u_1(a)t + u_2(a)t^2 + \cdots \in A[[t]]$.

- *If u satisfies $u(a_1 a_2)u(a_3) = u(a_1)u(a_2 a_3)$ mod t^2, then $u_1 \in Z^2_{S^2}(A, A)$.*

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- II *If u satisfies $u(a_1a_2)u(a_3) = u(a_1)u(a_2a_3)$ mod t^{n+1}, then we can extend u so that it satisfies $u(a_1a_2)u(a_3) = u(a_1)u(a_2a_3)$ mod t^{n+2} if and only if*

$$\sum_{i+j=n+1} u_i \circ u_j = 0 \in \mathrm{H}^3_{S^2}(A, A).$$

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For d odd

$$u(a_1 a_2) \cdots u(a_d a_{d+1}) = u(a_1)u(a_2 a_3) \cdots u(a_{d-1} a_d)u(a_{d+1}). \quad (1)$$

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$$\sum_{m=2}^{\lceil \frac{d+2}{2} \rceil} \left(\sum_{i_1+\cdots+i_m=n+1} u_{i_1} \circ \cdots \circ u_{i_m} \right) = 0 \in \mathrm{H}^{d+1}_{S^d}(A, A).$$

Observations and Questions

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Thank you!