The Derived Category

Jake Laubacher

September 29, 2015

Last Time

Theorem

$$Ext_R^n(A, B) \cong Hom_{\mathcal{D}(R-\mathbf{mod})}(\mathbf{A}[0], \mathbf{B}[n]).$$

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We still need to construct the derived category.

Categories

Definition

Two chain maps $f, g : \mathbf{C} \to \mathbf{D}$ are said to be *chain homotopic* if there exist a family of maps $\{s_n\}$ with $s_n : C_n \to D_{n+1}$ such that $f - g = ds + sc$.

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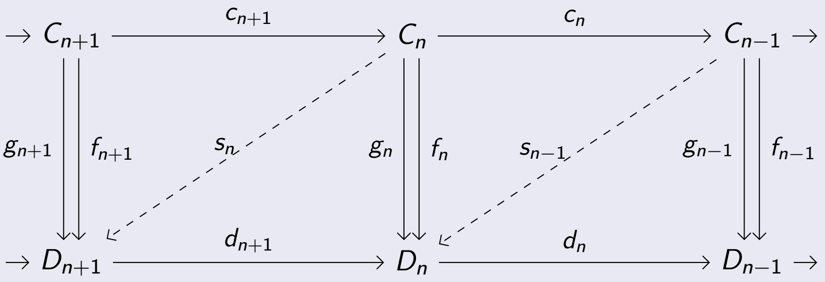

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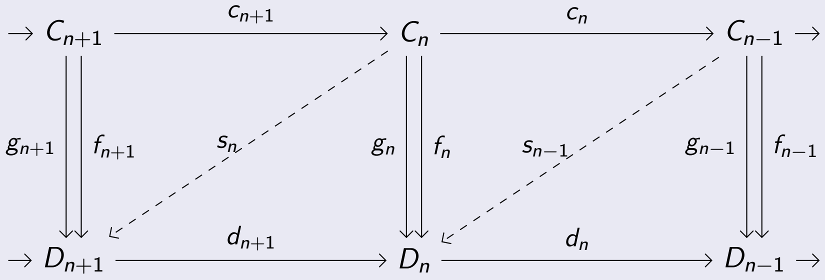

Theorem

Chain homotopies form an equivalence relation.

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$$\begin{array}{ccc} \exists W & \overset{\exists g}{\dashrightarrow} & Z \\ {\scriptstyle \exists\beta}\,\vdots & & \downarrow{\scriptstyle \alpha} \\ X & \xrightarrow{f} & Y. \end{array}$$

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Quasi-isomorphisms in $\mathbf{K}(R - \mathbf{mod})$ *form a multiplicative system.*

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Given a category $\mathcal{C}$ and a collection of morphisms S in $\mathcal{C}$, the *localization* of $\mathcal{C}$ *with respect to* S is a category $S^{-1}\mathcal{C}$ together with a functor

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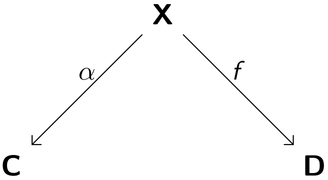

where **X** is a chain complex of R-modules, α is a quasi-isomorphism, and f is a chain map.

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Theorem (Gabriel-Zisman)

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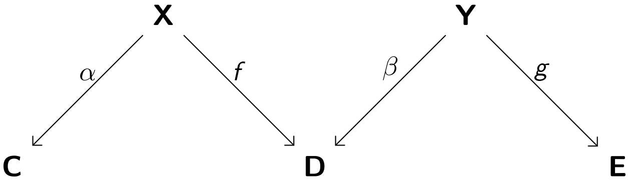

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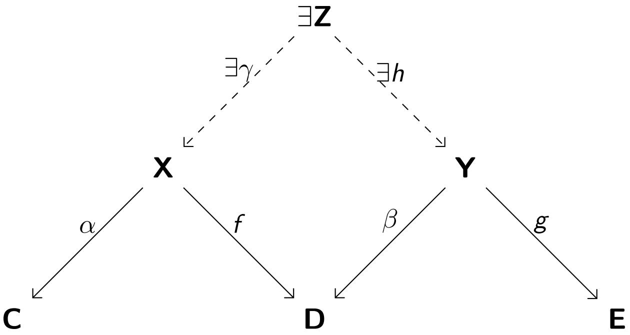

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We call a roof $f\alpha^{-1} := (f, \alpha)$ equivalent to the roof $g\beta^{-1} := (g, \beta)$ iff there exists δ and γ such that $(f\gamma, \alpha\gamma) = (g\delta, \beta\delta)$. In other words, it makes the following diagram commute:

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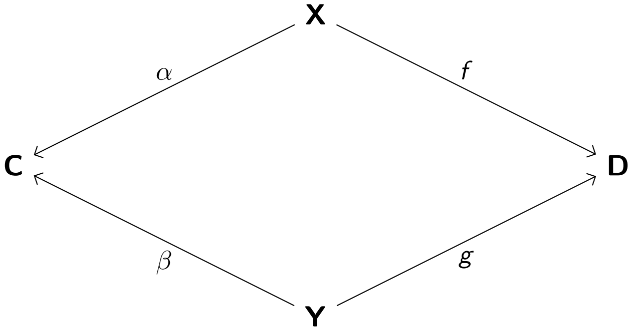

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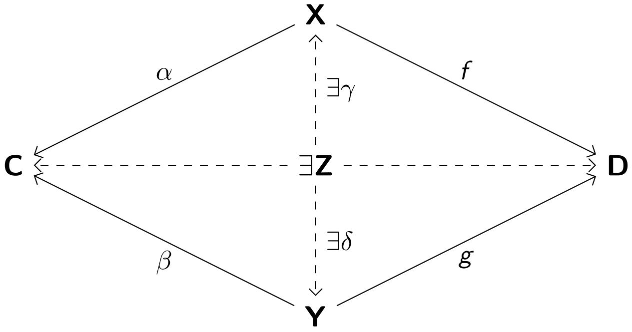

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Lemma 0

If two maps $f, g : \mathbf{C} \to \mathbf{D}$ in $\mathbf{K}(R - \mathbf{mod})$ become identified in $Q^{-1}\mathbf{K}(R - \mathbf{mod}) = \mathcal{D}(R - \mathbf{mod})$, then $f\alpha = g\alpha$ for some $\alpha \in Q$.

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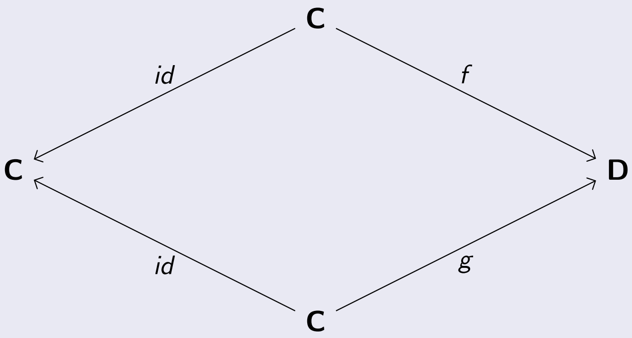

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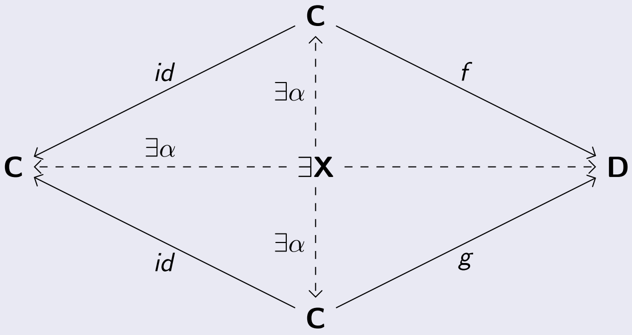

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Let $\mathbf{P}$ be a chain complex of projectives. For every quasi-isomorphism $\alpha : \mathbf{X} \to \mathbf{P}$, there exists $h : \mathbf{P} \to \mathbf{X}$ in $\mathbf{K}(R - \mathbf{mod})$ such that $\alpha h = id_{\mathbf{P}}$.

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- Take $\alpha \in Hom_{\mathbf{K}(R-\mathbf{mod})}(\mathbf{P}, \mathbf{B}[n])$. Then $\alpha \mapsto \alpha_n$.
- $\alpha_n \circ d_{n+1} = 0 \implies d_{n+1}^*(\alpha_n) = 0 \implies \alpha_n \in Ker(d_{n+1}^*)$.

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- $\alpha_n \circ d_{n+1} = 0 \implies d^*_{n+1}(\alpha_n) = 0 \implies \alpha_n \in Ker(d^*_{n+1})$.
- If α is chain homotopic to β, then

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$$\alpha_n - \beta_n = s_{n-1} d_n.$$

- $\alpha_n = \beta_n + s_{n-1} d_n = \beta_n + d^*_n(s_{n-1})$. Thus

$$\alpha_n \in \frac{Ker(d^*_{n+1})}{Im(d^*_n)} = Ext^n(A, B).$$

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- $\alpha_n \in Ext^n(A, B)$ corresponds to a chain map α, which is α_n in dimension n and zero elsewhere.

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Proof.

- $\alpha_n \in Ext^n(A,B)$ corresponds to a chain map α, which is α_n in dimension n and zero elsewhere.
- Conversely, say chain maps α and β both map to the same class. Then α and β are chain homotopic since

$$\beta_n - \alpha_n = s_{n-1}d_n.$$

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Consider the roof $f\alpha^{-1} : \mathbf{P} \leftarrow \mathbf{X} \rightarrow \mathbf{D}$. Note that by Lemma 2, there exists $h : \mathbf{P} \rightarrow \mathbf{X}$ in $\mathbf{K}(R - \mathbf{mod})$ such that $\alpha h = id_{\mathbf{P}}$.

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Conversely, if $f, g : \mathbf{P} \rightarrow \mathbf{D}$ are in $\mathbf{K}(R - \mathbf{mod})$ and become identified in the localization, then by Lemma 0 $f\beta = g\beta$ for some $\beta : \mathbf{Y} \rightarrow \mathbf{P}$.

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$$f = f \circ id_{\mathbf{P}} = f\beta k = g\beta k = g \circ id_{\mathbf{P}} = g.$$

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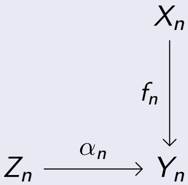
Proof.

We'll show part (*iii*). Consider the diagram below:



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Proof.

We'll show part (iii). Consider the diagram below:

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Lemma 2

Let $\mathbf{P}$ be a chain complex of projectives. For every quasi-isomorphism $\alpha : \mathbf{X} \to \mathbf{P}$, there exists $h : \mathbf{P} \to \mathbf{X}$ in $\mathbf{K}(R - \mathbf{mod})$ such that $\alpha h = id_{\mathbf{P}}$.

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- h is a chain map, so $h \in \mathbf{K}(R - \mathbf{mod})$.
- $\alpha h - id_{\mathbf{P}} = sd + ds$, so $\alpha h = id_{\mathbf{P}}$ in $\mathbf{K}(R - \mathbf{mod})$.

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Thank You!