# Simplicial Structures for Higher Order Hochschild Homology over the 2-Sphere

Jacob Laubacher

Saint Norbert College

AMS Eastern Sectional Conference
SUNY Buffalo, September 16, 2017

# Goal of the Talk

## Folklore

$$\mathrm{H}_\bullet(M \otimes_{A \otimes A^{op}} \mathcal{B}(A)) \cong \mathrm{H}_\bullet(A, M).$$

# Goal of the Talk

## Folklore

$$\mathrm{H}_\bullet(M \otimes_{A \otimes A^{op}} \mathcal{B}(A)) \cong \mathrm{H}_\bullet(A, M).$$

$$\mathrm{H}_\bullet(A, M) \cong \mathrm{H}_\bullet^{S^1}(A, M).$$

# Goal of the Talk

## Folklore

$$H_\bullet(M \otimes_{A\otimes A^{op}} \mathcal{B}(A)) \cong H_\bullet(A, M).$$

$$H_\bullet(A, M) \cong H^{S^1}_\bullet(A, M).$$

## Theorem

$$H_\bullet(\mathcal{M}^2(M) \otimes_{\mathcal{A}^2(A)} \mathcal{B}^2(A)) \cong H^{S^2}_\bullet(A, M).$$

- Fix $k$ to be a field, $\otimes = \otimes_k$, and all $k$-algebras have unit.

# Hochschild Homology

- Fix $k$ to be a field, $\otimes = \otimes_k$, and all $k$-algebras have unit.
- Let $A$ be a commutative $k$-algebra and $M$ an $A$-bimodule which is $A$-symmetric.

# Hochschild Homology

- Fix $k$ to be a field, $\otimes = \otimes_k$, and all $k$-algebras have unit.
- Let $A$ be a commutative $k$-algebra and $M$ an $A$-bimodule which is $A$-symmetric.

**Definition (Hochschild - 1945)**

The **Hochschild homology of** $A$ **with coefficients in** $M$ is denoted as $\mathrm{H}_\bullet(A, M)$ and is defined to be the homology of the chain complex

$$\dots \xrightarrow{d_{n+1}} M \otimes A^{\otimes n} \xrightarrow{d_n} M \otimes A^{\otimes n-1} \xrightarrow{d_{n-1}} \dots$$

$$\dots \xrightarrow{d_4} M \otimes A^{\otimes 3} \xrightarrow{d_3} M \otimes A^{\otimes 2} \xrightarrow{d_2} M \otimes A \xrightarrow{d_1} M \longrightarrow 0.$$

# Hochschild Cohomology

$$\dots \xrightarrow{d_{n+1}} M \otimes A^{\otimes n} \xrightarrow{d_n} M \otimes A^{\otimes n-1} \xrightarrow{d_{n-1}} \dots$$

$$\dots \xrightarrow{d_4} M \otimes A^{\otimes 3} \xrightarrow{d_3} M \otimes A^{\otimes 2} \xrightarrow{d_2} M \otimes A \xrightarrow{d_1} M \longrightarrow 0.$$

## Example

$$d_1(m \otimes a) = ma - am = 0.$$

# Hochschild Cohomology

$$\cdots \xrightarrow{d_{n+1}} M \otimes A^{\otimes n} \xrightarrow{d_n} M \otimes A^{\otimes n-1} \xrightarrow{d_{n-1}} \cdots$$

$$\cdots \xrightarrow{d_4} M \otimes A^{\otimes 3} \xrightarrow{d_3} M \otimes A^{\otimes 2} \xrightarrow{d_2} M \otimes A \xrightarrow{d_1} M \longrightarrow 0.$$

## Example

$$d_1(m \otimes a) = ma - am = 0.$$

$$d_2(m \otimes a \otimes b) = ma \otimes b - m \otimes ab + bm \otimes a.$$

# Hochschild Cohomology

$$\cdots \xrightarrow{d_{n+1}} M \otimes A^{\otimes n} \xrightarrow{d_n} M \otimes A^{\otimes n-1} \xrightarrow{d_{n-1}} \cdots$$

$$\cdots \xrightarrow{d_4} M \otimes A^{\otimes 3} \xrightarrow{d_3} M \otimes A^{\otimes 2} \xrightarrow{d_2} M \otimes A \xrightarrow{d_1} M \longrightarrow 0.$$

## Example

$$d_1(m \otimes a) = ma - am = 0.$$

$$d_2(m \otimes a \otimes b) = ma \otimes b - m \otimes ab + bm \otimes a.$$

$$d_3(m\otimes a\otimes b\otimes c) = ma\otimes b\otimes c - m\otimes ab\otimes c + m\otimes a\otimes bc - cm\otimes a\otimes b.$$

**Definition (Anderson - 1971, Pirashvili - 2000)**

The **higher order Hochschild homology of** $A$ **with values in** $M$ **over the simplicial set** $S^2$ is denoted as $\mathrm{H}_\bullet^{S^2}(A, M)$ and is defined to be the homology of the chain complex

$$\ldots \xrightarrow{\partial_{n+1}} M \otimes A^{\otimes \frac{n(n-1)}{2}} \xrightarrow{\partial_n} M \otimes A^{\otimes \frac{(n-1)(n-2)}{2}} \xrightarrow{\partial_{n-1}} \ldots$$

$$\ldots \xrightarrow{\partial_5} M \otimes A^{\otimes 6} \xrightarrow{\partial_4} M \otimes A^{\otimes 3} \xrightarrow{\partial_3} M \otimes A \xrightarrow{\partial_2} M \xrightarrow{\partial_1} M \longrightarrow 0.$$

# Higher Order Hochschild Homology

## Definition (Anderson - 1971, Pirashvili - 2000)

The **higher order Hochschild homology of** $A$ **with values in** $M$ **over the simplicial set** $S^2$ is denoted as $\mathrm{H}_\bullet^{S^2}(A,M)$ and is defined to be the homology of the chain complex

$$\dots \xrightarrow{\partial_{n+1}} M\otimes A^{\otimes \frac{n(n-1)}{2}} \xrightarrow{\partial_n} M\otimes A^{\otimes \frac{(n-1)(n-2)}{2}} \xrightarrow{\partial_{n-1}} \dots$$

$$\dots \xrightarrow{\partial_5} M\otimes A^{\otimes 6} \xrightarrow{\partial_4} M\otimes A^{\otimes 3} \xrightarrow{\partial_3} M\otimes A \xrightarrow{\partial_2} M \xrightarrow{\partial_1} M \longrightarrow 0.$$

## Example

$$\partial_1(m) = m - m = 0.$$

$$\partial_2(m\otimes a) = ma - ma + ma = ma.$$

$$\cdots \xrightarrow{\partial_5} M \otimes A^{\otimes 6} \xrightarrow{\partial_4} M \otimes A^{\otimes 3} \xrightarrow{\partial_3} M \otimes A \xrightarrow{\partial_2} M \xrightarrow{\partial_1} M \longrightarrow 0.$$

### EXAMPLE

$$\partial_3 \left( m \otimes \begin{pmatrix} a & b \\ 1 & c \end{pmatrix} \right) = mab \otimes c - ma \otimes bc + mc \otimes ab - mbc \otimes a.$$

$$\dots \xrightarrow{\partial_5} M \otimes A^{\otimes 6} \xrightarrow{\partial_4} M \otimes A^{\otimes 3} \xrightarrow{\partial_3} M \otimes A \xrightarrow{\partial_2} M \xrightarrow{\partial_1} M \longrightarrow 0.$$

**Example**

$$\partial_3 \left( m \otimes \begin{pmatrix} a & b \\ 1 & c \end{pmatrix} \right) = mab \otimes c - ma \otimes bc + mc \otimes ab - mbc \otimes a.$$

$$\partial_4 \left( m \otimes \begin{pmatrix} a & b & c \\ 1 & d & e \\ 1 & 1 & f \end{pmatrix} \right) = mabc \otimes \begin{pmatrix} d & e \\ 1 & f \end{pmatrix} - ma \otimes \begin{pmatrix} bd & ce \\ 1 & f \end{pmatrix}$$
$$+ md \otimes \begin{pmatrix} ab & c \\ 1 & ef \end{pmatrix} - mf \otimes \begin{pmatrix} a & bc \\ 1 & de \end{pmatrix} + mcef \otimes \begin{pmatrix} a & b \\ 1 & d \end{pmatrix}.$$

# Motivation

**Remark (Hochschild - 1945, Gerstenhaber - 1964)**

Hochschild cohomology can be used to study extensions of algebras over a field. It also has been exploited in the study of deformations of algebras.

# Motivation

**Remark (Hochschild - 1945, Gerstenhaber - 1964)**

Hochschild cohomology can be used to study extensions of algebras over a field. It also has been exploited in the study of deformations of algebras.

**Remark (Connes - 1985)**

Hochschild homology leads directly to cyclic homology.

**REMARK (HOCHSCHILD - 1945, GERSTENHABER - 1964)**

Hochschild cohomology can be used to study extensions of algebras over a field. It also has been exploited in the study of deformations of algebras.

**REMARK (CONNES - 1985)**

Hochschild homology leads directly to cyclic homology.

**REMARK (GINOT, TRADLER, AND ZEINALIAN - 2014)**

Higher order Hochschild (co)homology applies to brane and string topology.

# Goal of the Talk

**Theorem**

$$\mathrm{H}_\bullet(M \otimes_{A\otimes A^{op}} \mathcal{B}(A)) \cong \mathrm{H}_\bullet(A, M) \cong \mathrm{H}_\bullet^{S^1}(A, M).$$

$$\mathrm{H}_\bullet(\mathcal{M}^2(M) \otimes_{\mathcal{A}^2(A)} \mathcal{B}^2(A)) \cong \mathrm{H}_\bullet^{S^2}(A, M).$$

# Goal of the Talk

**Theorem**

$$\mathrm{H}_\bullet(M \otimes_{A\otimes A^{op}} \mathcal{B}(A)) \cong \mathrm{H}_\bullet(A, M) \cong \mathrm{H}_\bullet^{S^1}(A, M).$$

$$\mathrm{H}_\bullet(\mathcal{M}^2(M) \otimes_{\mathcal{A}^2(A)} \mathcal{B}^2(A)) \cong \mathrm{H}_\bullet^{S^2}(A, M).$$

We'll study the simplicial structure of the complexes associated to:

# Goal of the Talk

**Theorem**

$$\mathrm{H}_\bullet(M \otimes_{A\otimes A^{op}} \mathcal{B}(A)) \cong \mathrm{H}_\bullet(A, M) \cong \mathrm{H}_\bullet^{S^1}(A, M).$$

$$\mathrm{H}_\bullet(\mathcal{M}^2(M) \otimes_{\mathcal{A}^2(A)} \mathcal{B}^2(A)) \cong \mathrm{H}_\bullet^{S^2}(A, M).$$

We'll study the simplicial structure of the complexes associated to:

- the Hochschild homology of $A$ with coefficients in $M$, $\mathrm{H}_\bullet(A, M)$.

# Goal of the Talk

**Theorem**

$$\mathrm{H}_\bullet(M \otimes_{A\otimes A^{op}} \mathcal{B}(A)) \cong \mathrm{H}_\bullet(A, M) \cong \mathrm{H}_\bullet^{S^1}(A, M).$$

$$\mathrm{H}_\bullet(\mathcal{M}^2(M) \otimes_{\mathcal{A}^2(A)} \mathcal{B}^2(A)) \cong \mathrm{H}_\bullet^{S^2}(A, M).$$

We'll study the simplicial structure of the complexes associated to:

- the Hochschild homology of $A$ with coefficients in $M$, $\mathrm{H}_\bullet(A, M)$.
- the higher order Hochschild homology of $A$ with values in $M$ over the simplicial set $S^2$, $\mathrm{H}_\bullet^{S^2}(A, M)$.

# Simplicial Identities

Recall the following: For

$$\delta_i : X_n \longrightarrow X_{n-1}$$

and

$$\sigma_i : X_n \longrightarrow X_{n+1},$$

where $0 \leq i \leq n$, we have:

# Simplicial Identities

Recall the following: For

$$\delta_i : X_n \longrightarrow X_{n-1}$$

and

$$\sigma_i : X_n \longrightarrow X_{n+1},$$

where $0 \leq i \leq n$, we have:

## The Simplicial Identities

$$\begin{aligned}
\delta_i\delta_j &= \delta_{j-1}\delta_i \quad \text{if} \quad i < j,\\
\sigma_i\sigma_j &= \sigma_{j+1}\sigma_i \quad \text{if} \quad i \leq j,\\
\delta_i\sigma_j &= \sigma_{j-1}\delta_i \quad \text{if} \quad i < j,\\
\delta_i\sigma_j &= \mathrm{id} \quad \text{if} \quad i = j \quad \text{or} \quad i = j+1,\\
\delta_i\sigma_j &= \sigma_j\delta_{i-1} \quad \text{if} \quad i > j+1.
\end{aligned}$$

### Definition

A **simplicial $k$-algebra** is a collection of $k$-algebras $A_n$ together with morphisms of $k$-algebras

$$\delta_i^{\mathcal{A}} : A_n \longrightarrow A_{n-1} \quad \text{and} \quad \sigma_i^{\mathcal{A}} : A_n \longrightarrow A_{n+1}$$

such that the simplicial identities are satisfied.

### Definition

A **simplicial $k$-algebra** is a collection of $k$-algebras $A_n$ together with morphisms of $k$-algebras

$$\delta_i^{\mathcal{A}} : A_n \longrightarrow A_{n-1} \quad \text{and} \quad \sigma_i^{\mathcal{A}} : A_n \longrightarrow A_{n+1}$$

such that the simplicial identities are satisfied.

### Example

We define the simplicial $k$-algebra $\mathcal{A}(A \otimes A^{op})$ by setting $A_n = A \otimes A^{op}$, $\delta_i^{\mathcal{A}} = \mathrm{id}_{A \otimes A^{op}}$, and $\sigma_i^{\mathcal{A}} = \mathrm{id}_{A \otimes A^{op}}$.

### Example

$\mathcal{A}^2(A)$ is a simplicial $k$-algebra by setting $A_n = A^{\otimes 3n+3}$ for all $n$.

$$\delta_1^{\mathcal{A}}\left(\otimes\begin{pmatrix} a_0 & b_1 & b_2 & b_3 & d \\ & a_1 & & & c_1 \\ & & a_2 & & c_2 \\ & & & a_3 & c_3 \\ & & & & a_4 \end{pmatrix}\right) = \otimes\begin{pmatrix} a_0 & b_1 b_2 & b_3 & d \\ & a_1 a_2 & & c_1 c_2 \\ & & a_3 & c_3 \\ & & & a_4 \end{pmatrix},$$

and $\sigma_i^{\mathcal{A}}$ is defined similarly.

# Simplicial Left Modules

## Definition

We say that $\mathcal{M}$ is a **simplicial left module** over a simplicial $k$-algebra $\mathcal{A}$ if $\mathcal{M} = (M_n)_{n\geq 0}$ is a simplicial $k$-vector space together with a left $A_n$-module structure on $M_n$ for all $n \geq 0$ such that we have the following natural compatibility conditions:

$$\delta_i^{\mathcal{M}}(a_n m_n) = \delta_i^{\mathcal{A}}(a_n)\delta_i^{\mathcal{M}}(m_n) \quad \text{and} \quad \sigma_i^{\mathcal{M}}(a_n m_n) = \sigma_i^{\mathcal{A}}(a_n)\sigma_i^{\mathcal{M}}(m_n)$$

for all $a_n \in A_n$, for all $m_n \in M_n$, and for $0 \leq i \leq n$.

# Simplicial Left Modules

## Definition

We say that $\mathcal{M}$ is a **simplicial left module** over a simplicial $k$-algebra $\mathcal{A}$ if $\mathcal{M} = (M_n)_{n\geq 0}$ is a simplicial $k$-vector space together with a left $A_n$-module structure on $M_n$ for all $n \geq 0$ such that we have the following natural compatibility conditions:

$$\delta_i^{\mathcal{M}}(a_n m_n) = \delta_i^{\mathcal{A}}(a_n)\delta_i^{\mathcal{M}}(m_n) \quad \text{and} \quad \sigma_i^{\mathcal{M}}(a_n m_n) = \sigma_i^{\mathcal{A}}(a_n)\sigma_i^{\mathcal{M}}(m_n)$$

for all $a_n \in A_n$, for all $m_n \in M_n$, and for $0 \leq i \leq n$.

## Remark

We can define **simplicial right modules** in a similar way.

# Simplicial Left Modules

### Definition

We say that $\mathcal{M}$ is a **simplicial left module** over a simplicial $k$-algebra $\mathcal{A}$ if $\mathcal{M} = (M_n)_{n \geq 0}$ is a simplicial $k$-vector space together with a left $A_n$-module structure on $M_n$ for all $n \geq 0$ such that we have the following natural compatibility conditions:

$$\delta_i^{\mathcal{M}}(a_n m_n) = \delta_i^{\mathcal{A}}(a_n)\delta_i^{\mathcal{M}}(m_n) \quad \text{and} \quad \sigma_i^{\mathcal{M}}(a_n m_n) = \sigma_i^{\mathcal{A}}(a_n)\sigma_i^{\mathcal{M}}(m_n)$$

for all $a_n \in A_n$, for all $m_n \in M_n$, and for $0 \leq i \leq n$.

### Remark

We can define **simplicial right modules** in a similar way.

### Example

We define the simplicial left module $\mathcal{B}(A)$ over the simplicial $k$-algebra $\mathcal{A}(A \otimes A^{op})$ by setting $B_n = A^{\otimes n+2}$ for all $n \geq 0$.

### Example

$\mathcal{B}^2(A)$ is a simplicial left module over $\mathcal{A}^2(A)$ by setting $B_n = A^{\otimes \frac{(n+2)(n+3)}{2}}$ for all $n \geq 0$. The left module structure:

$$\otimes \begin{pmatrix} a_0 & b_1 & b_2 & b_3 & d \\ & a_1 & & & c_1 \\ & & a_2 & & c_2 \\ & & & a_3 & c_3 \\ & & & & a_4 \end{pmatrix} \cdot \otimes \begin{pmatrix} a_{0,0} & a_{0,1} & a_{0,2} & a_{0,3} & a_{0,4} \\ 1 & a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ 1 & 1 & a_{2,2} & a_{2,3} & a_{2,4} \\ 1 & 1 & 1 & a_{3,3} & a_{3,4} \\ 1 & 1 & 1 & 1 & a_{4,4} \end{pmatrix}$$

$$= \otimes \begin{pmatrix} a_0 a_{0,0} & b_1 a_{0,1} & b_2 a_{0,2} & b_3 a_{0,3} & d a_{0,4} \\ 1 & a_1 a_{1,1} & a_{1,2} & a_{1,3} & c_1 a_{1,4} \\ 1 & 1 & a_2 a_{2,2} & a_{2,3} & c_2 a_{2,4} \\ 1 & 1 & 1 & a_3 a_{3,3} & c_3 a_{3,4} \\ 1 & 1 & 1 & 1 & a_4 a_{4,4} \end{pmatrix}.$$

# Simplicial Right Modules

**Example**

$\mathcal{M}(M)$ is a simplicial right module over $\mathcal{A}(A \otimes A^{op})$ by setting $M_n = M$, $\delta_i^{\mathcal{M}} = \mathrm{id}_M$, $\sigma_i^{\mathcal{M}} = \mathrm{id}_M$, and $m \cdot (a \otimes b) = bma$.

# Simplicial Right Modules

### Example

$\mathcal{M}(M)$ is a simplicial right module over $\mathcal{A}(A \otimes A^{op})$ by setting $M_n = M$, $\delta_i^{\mathcal{M}} = \mathrm{id}_M$, $\sigma_i^{\mathcal{M}} = \mathrm{id}_M$, and $m \cdot (a \otimes b) = bma$.

### Example

$\mathcal{M}^2(M)$ is a simplicial right module over $\mathcal{A}^2(A)$ by setting $M_n = M$ for all $n \geq 0$. The right module structure is

$$m \cdot \otimes \begin{pmatrix} a_0 & b_1 & b_2 & b_3 & d \\ & a_1 & & & c_1 \\ & & a_2 & & c_2 \\ & & & a_3 & c_3 \\ & & & & a_4 \end{pmatrix} = ma_0a_1a_2a_3a_4b_1b_2b_3c_1c_2c_3d.$$

Again, $\delta_i^{\mathcal{M}} = \mathrm{id}_M$ and $\sigma_i^{\mathcal{M}} = \mathrm{id}_M$.

# Results

## Folklore

$$\mathrm{H}_\bullet(M \otimes_{A\otimes A^{op}} \mathcal{B}(A)) \cong \mathrm{H}_\bullet(\mathcal{M}(M) \otimes_{\mathcal{A}(A\otimes A^{op})} \mathcal{B}(A)) \cong \mathrm{H}_\bullet^{S^1}(A, M).$$

# Results

## Folklore

$$\mathrm{H}_\bullet(M \otimes_{A\otimes A^{op}} \mathcal{B}(A)) \cong \mathrm{H}_\bullet(\mathcal{M}(M) \otimes_{\mathcal{A}(A\otimes A^{op})} \mathcal{B}(A)) \cong \mathrm{H}_\bullet^{S^1}(A, M).$$

## Lemma (L, Staic, and Stancu - 2016)

*Suppose that $(\mathcal{X}, \delta_i^{\mathcal{X}}, \sigma_i^{\mathcal{X}})$ is a simplicial right module over the simplicial k-algebra $\mathcal{A}$, and $(\mathcal{Y}, \delta_i^{\mathcal{Y}}, \sigma_i^{\mathcal{Y}})$ is a simplicial left module over the simplicial k-algebra $\mathcal{A}$. Then $(\mathcal{X} \otimes_{\mathcal{A}} \mathcal{Y}, D_i, S_i)$ is a simplicial k-module.*

# Results

## Folklore

$$\mathrm{H}_\bullet(M \otimes_{A\otimes A^{op}} \mathcal{B}(A)) \cong \mathrm{H}_\bullet(\mathcal{M}(M) \otimes_{\mathcal{A}(A\otimes A^{op})} \mathcal{B}(A)) \cong \mathrm{H}_\bullet^{S^1}(A, M).$$

## Lemma (L, Staic, and Stancu - 2016)

*Suppose that $(\mathcal{X}, \delta_i^{\mathcal{X}}, \sigma_i^{\mathcal{X}})$ is a simplicial right module over the simplicial k-algebra $\mathcal{A}$, and $(\mathcal{Y}, \delta_i^{\mathcal{Y}}, \sigma_i^{\mathcal{Y}})$ is a simplicial left module over the simplicial k-algebra $\mathcal{A}$. Then $(\mathcal{X} \otimes_{\mathcal{A}} \mathcal{Y}, D_i, S_i)$ is a simplicial k-module.*

## Theorem

$$\mathcal{M}^2(M) \otimes_{\mathcal{A}^2(A)} \mathcal{B}^2(A)$$

*is a simplicial k-module, and*

$$\mathrm{H}_\bullet(\mathcal{M}^2(M) \otimes_{\mathcal{A}^2(A)} \mathcal{B}^2(A)) \cong \mathrm{H}_\bullet^{S^2}(A, M).$$

# Thank You!

Questions?