# Ramsey Numbers

Jacob Laubacher

SNC Mathematics Colloquium Series
September 7, 2017

### The Party Problem (Gleason and Greenwood - 1955)

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### Answer

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Faith Charlie

Elise Dean

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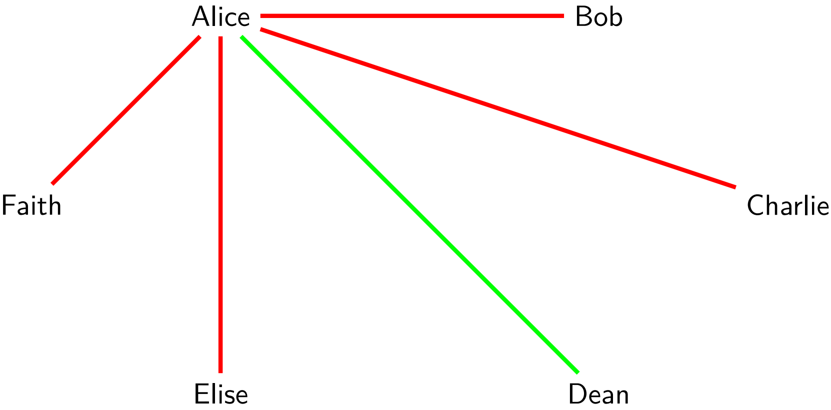

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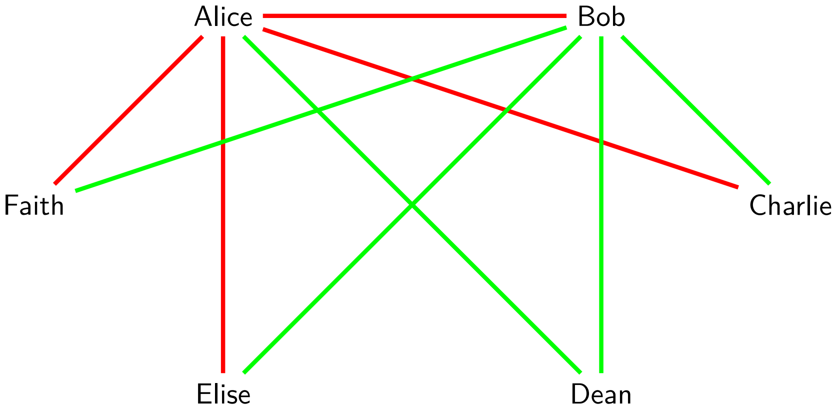

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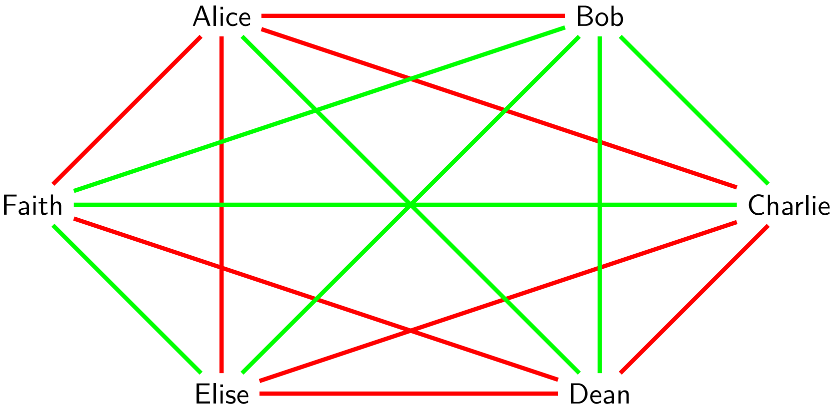

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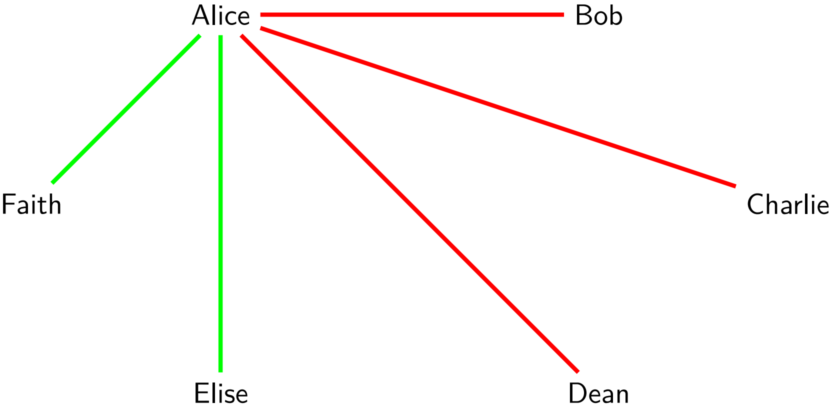

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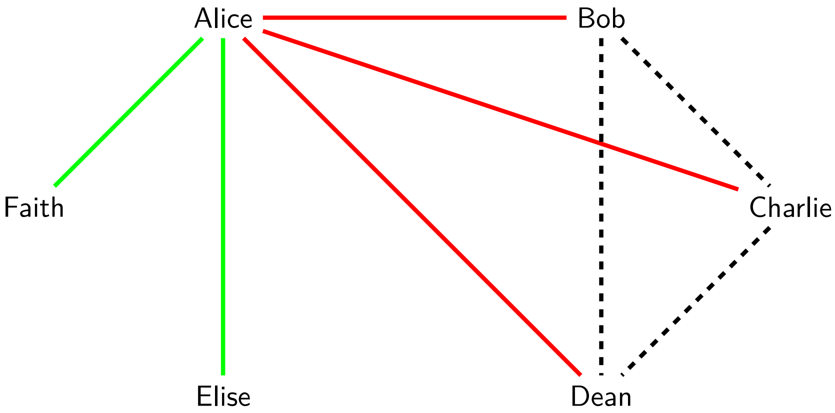

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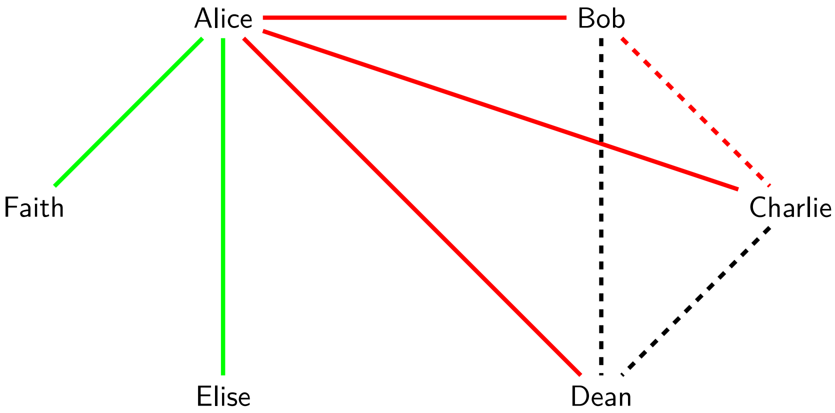

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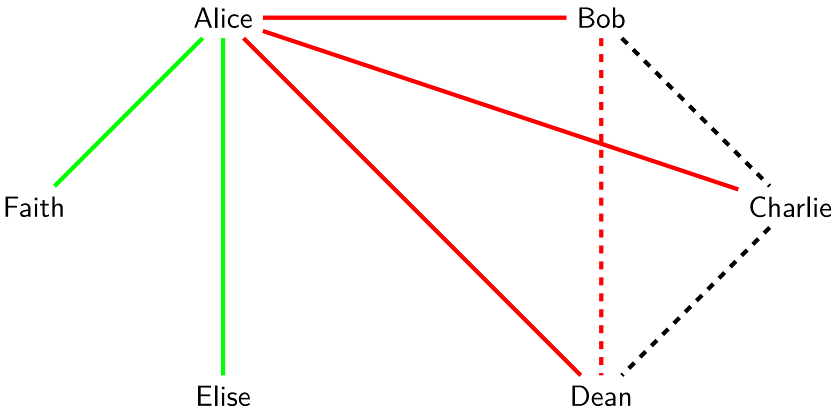

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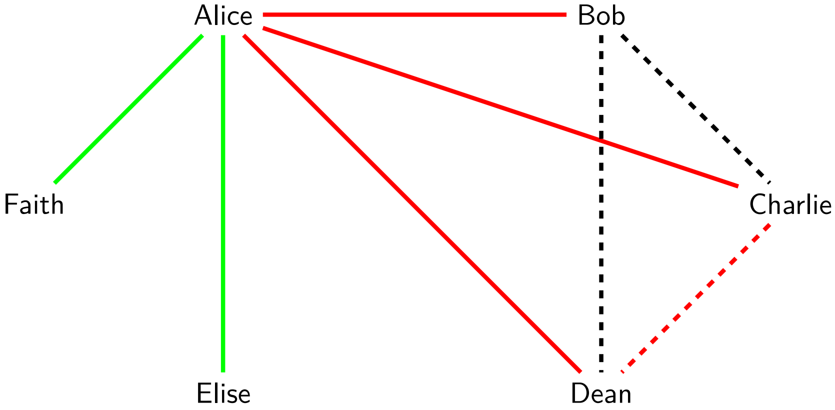

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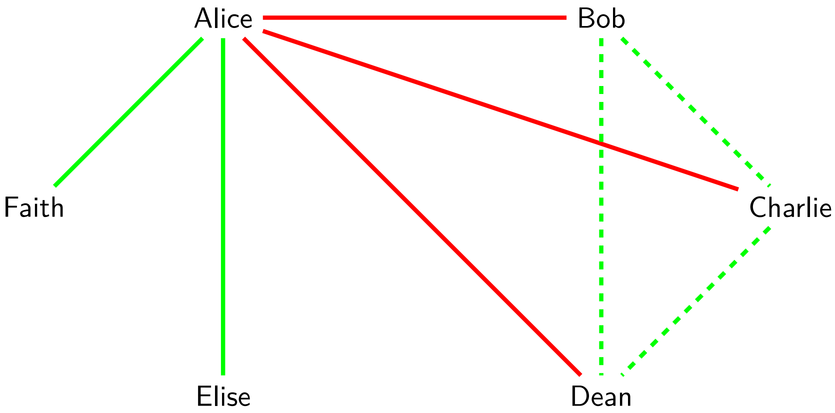

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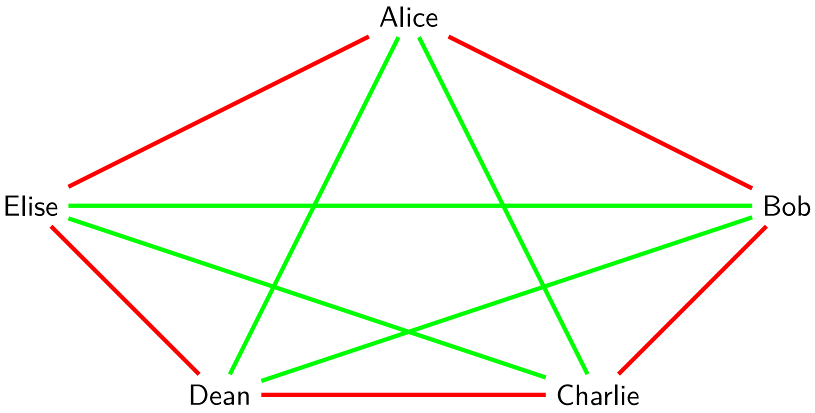

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3. $R(r, 2) = r$,
4. $R(r, s) \leq R(r-1, s) + R(r, s-1)$, *and*
5. *if both* $R(r-1, s)$ *and* $R(r, s-1)$ *are even, then* $R(r, s) \leq R(r-1, s) + R(r, s-1) - 1$.

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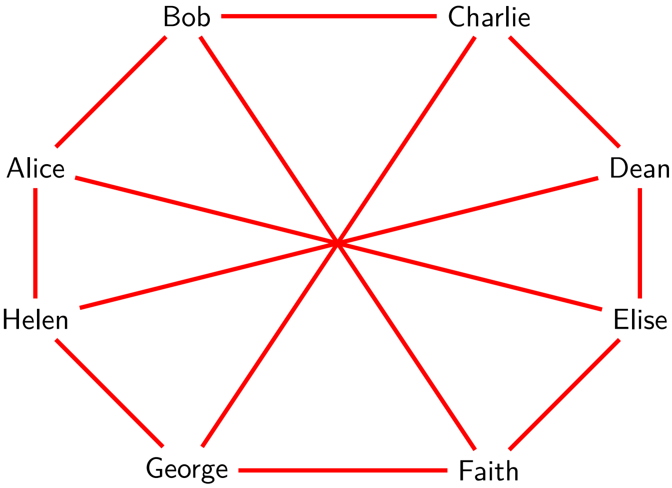

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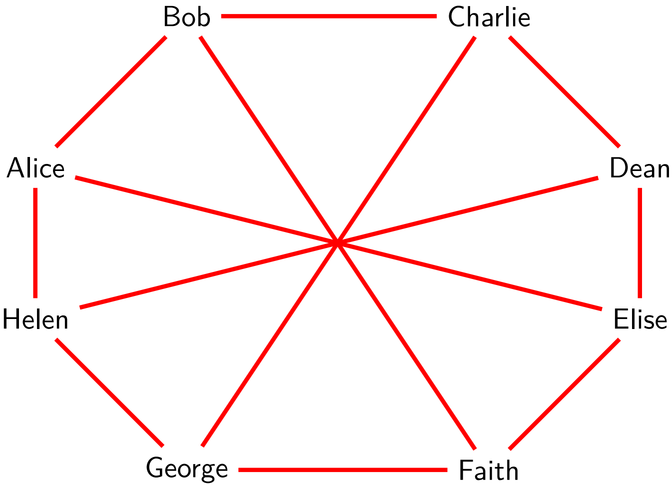


So $R(4,3) = 9$ (Gleason and Greenwood - 1955).

# Some Others

- One can get

$$R(5,3) \leq R(4,3) + R(5,2) = 9 + 5 = 14,$$

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- A whole lot of work to get $R(5,4) = 25$ (McKay and Radziszowski - 1995).

# A Story

## Sandor Szalai

"In the course of an examination of friendship between children some fifty years ago, the Hungarian sociologist Sandor Szalai observed that among any group of about twenty children he checked he could always find four children any two of whom were friends, or else four children no two of whom were friends. Despite the temptation to try to draw sociological conclusions, Szalai realized that this might well be a mathematical phenomenon rather than sociological one. Indeed, a brief discussion with the mathematicians Erdös, Turán, and Sós convinced him this was the case."

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Recently in March, the bound was made smaller (by Angeltveit and McKay - 2017) to

$$43 \leq R(5,5) \leq 48.$$

# $R(5,5)$?

**Paul Erdös - 1993**

"Suppose aliens invade the Earth and threaten to destroy it in a year if human beings do not find $R(5,5)$. It is (probably) possible to save the Earth by putting together the world's best minds and computers. If, however, the invaders were to demand $R(6,6)$, the human beings might as well attempt a preemptive strike without even trying to ponder the problem."

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So $R(5,5)$ seems possible.

But $R(6,6)$ currently has the bound

$$102 \leq R(6,6) \leq 165.$$

Not easy.

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There are bounds

$$30 \leq R(3,3,4) \leq 31,$$

but it was "shown" recently that $R(3,3,4) = 30$.

# Questions

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GMS 2121

Projects