The Bar Resolution and Generalizations

Preliminary Exam

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Introduction

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If A is a k-algebra and M is an A-bimodule, then

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- Hochschild Cohomology is motivated by Deformation Theory.
- Permits us to study the k-algebra structure on $A[[t]]$.

Rings and Modules

Definition

A **ring** R is a nonempty set with two binary operations $+$ and $\cdot$ (addition and multiplication, respectively) such that for all $a, b, c \in R$:

(i) $a + b = b + a$.

(ii) $(a + b) + c = a + (b + c)$.

(iii) There is an additive identity 0. That is, $a + 0 = a$.

(iv) There are additive inverses $-a$. Thus, $a + (-a) = 0$.

(v) $(ab)c = a(bc)$.

(vi) $a(b + c) = ab + ac$.

(vii) $(a + b)c = ac + bc$.

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(vii) $(a + b)c = ac + bc$.

Example

$(\mathbb{Z}, +, \cdot)$, $(R[x], +, \cdot)$, $(M_n(R), +, \cdot)$

Definition

A **left module** over R is a set M consisting of an abelian group $(M, +)$ along with the operation $\cdot : R \times M \to M$ with $(r, m) \mapsto r \cdot m$ such that for all $r, s \in R$ and for all $m, n \in M$ we have that

(i) $r \cdot (m + n) = r \cdot m + r \cdot n$,

(ii) $(r + s) \cdot m = r \cdot m + s \cdot m$,

(iii) $r \cdot (s \cdot m) = (rs) \cdot m$, and

(iv) $1_R \cdot m = m$.

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Example

- Modules over a field are vector spaces.
- $\mathbb{Z}$-modules are abelian groups.
- $\mathbb{C}$ is a $\mathbb{C}[x]$-module with $\cdot$ given by evaluation.
- R^n is an $M_n(R)$-module with $\cdot$ given by matrix multiplication.

Definition

Given two R-modules M and N, we define a **module homomorphism** by

$$\varphi : M \to N$$

such that

(i) $\varphi(m_1 + m_2) = \varphi(m_1) + \varphi(m_2)$, and

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Definition

If $\varphi : M \to N$ is a module homomorphism, then the **kernel** is given as

$$\mathrm{Ker}(\varphi) = \{m \in M \mid \varphi(m) = 0\}.$$

Similarly, the **image** is

$$\mathrm{Im}(\varphi) = \{n \in N \mid n = \varphi(m) \text{ for some } m \in M\}.$$

Exactness

Definition

A chain of modules

$$\cdots \xrightarrow{d_{n+2}} C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \xrightarrow{d_{n-2}} \cdots$$

is a **complex** if $\mathrm{Im}(d_{n+1}) \subseteq \mathrm{Ker}(d_n)$ for every n.

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A chain complex is **exact** at C_n if $\mathrm{Im}(d_{n+1}) = \mathrm{Ker}(d_n)$. So $\mathrm{H}_n(\mathbf{C}) = 0$.

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Definition

A chain complex is **exact** at C_n if $\mathrm{Im}(d_{n+1}) = \mathrm{Ker}(d_n)$. So $\mathrm{H}_n(\mathbf{C}) = 0$. Say that $\mathbf{C}$ is **exact** if $\mathrm{H}_n(\mathbf{C}) = 0$ for all n.

Exactness

Example

The chain complex

$$\cdots \xrightarrow{\cdot 4} \mathbb{Z}_8 \xrightarrow{\cdot 4} \mathbb{Z}_8 \xrightarrow{\cdot 4} \mathbb{Z}_8 \xrightarrow{\cdot 4} \mathbb{Z}_8 \xrightarrow{\cdot 4} \cdots$$

is not exact in any dimension.

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Example

The chain complex

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 5} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}_5 \longrightarrow 0$$

is exact.

Projective Modules

Definition

A module P is called **projective** if for any modules B and C with a surjective module homomorphism $g : B \to C$ and any module homomorphism $\gamma : P \to C$, there exists $\beta : P \to B$ such that $\gamma = g \circ \beta$.

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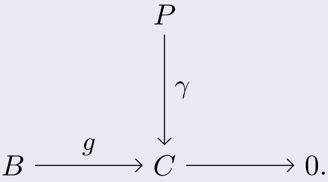

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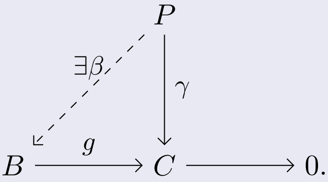

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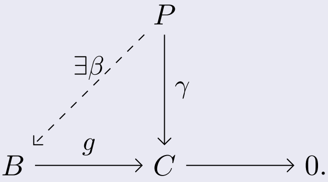

Example

All free modules are projective.

Projective Resolutions

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We define a **projective resolution** of an R-module M as a chain complex $\mathbf{P}$ together with a surjective module homomorphism $\varepsilon : P_0 \to M$ such that

$$\cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

is exact, and P_n is projective for all $n \geq 0$.

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Theorem

Every R-module has a projective resolution.

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We now recall the construction of the Ext groups, $\mathrm{Ext}^*_R(A, B)$.

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Theorem

Ext *does not depend on the projective resolution of A.*

k-algebras

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Example

$\mathbb{R}[x]$ is an $\mathbb{R}$-algebra with φ the inclusion map.

Bimodules

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- A-bimodules are equivalent to left $A^e = A \otimes A^{op}$-modules.

Hochschild Cohomology

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For example,

$$d_1(f)(a \otimes b) = af(b) - f(ab) + f(a)b$$

for $f \in \mathrm{Hom}_k(A, M)$.

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We have

$$d_2(g)(a \otimes b \otimes c) \\ = ag(b \otimes c) - g(ab \otimes c) + g(a \otimes bc) - g(a \otimes b)c$$

for $g \in \mathrm{Hom}_k(A^{\otimes 2}, M)$.

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Define

$$d_n : \mathrm{Hom}_k(A^{\otimes n}, M) \to \mathrm{Hom}_k(A^{\otimes n+1}, M)$$

by

$$\begin{aligned} d_n(f)(a_0 \otimes a_1 \otimes \ldots \otimes a_n) &= a_0 f(a_1 \otimes \ldots \otimes a_n) \\ &+ \sum_{i=1}^{n} (-1)^i f(a_0 \otimes \ldots \otimes a_{i-1}a_i \otimes \ldots \otimes a_n) \\ &+ (-1)^{n+1} f(a_0 \otimes \ldots \otimes a_{n-1})a_n. \end{aligned}$$

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Let $\mathbf{C}^*(A, M)$ denote the chain above.

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Lemma

$\mathbf{C}^*(A, M)$ is a chain complex.

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Lemma

$\mathbf{C}^*(A, M)$ *is a chain complex.*

Definition

The **Hochschild cohomology** of A with coefficients in M is denoted as $\mathrm{H}^*(A, M)$ and is defined to be the k-modules of the form

$$\mathrm{H}^n(A, M) = \mathrm{H}^n(\mathbf{C}^*(A, M)) = \frac{\mathrm{Ker}(d_n)}{\mathrm{Im}(d_{n-1})}.$$

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For example, we have:

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Denote this chain by $\mathbf{Bar}_*(A)$.

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Proof Sketch

One can check $\mathbf{Bar}_*(A)$ is a complex.
One can check $\mathbf{Bar}_*(A)$ is exact.

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One can check $\mathbf{Bar}_*(A)$ is a complex.
One can check $\mathbf{Bar}_*(A)$ is exact.
Projective because

$$A^{\otimes n+2} \cong A \otimes A^{\otimes n} \otimes A \cong A \otimes A^{op} \otimes A^{\otimes n} = A^e \otimes A^{\otimes n}.$$

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The Bar Resolution

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& \| & & \| & & \| & \\
& \mathrm{Hom}_k(A^{\otimes 2}, M) & & \mathrm{Hom}_k(A, M) & & \mathrm{Hom}_k(k, M) & \\
& \| & & \| & & \| & \\
& \mathbf{C}^2(A, M) & & \mathbf{C}^1(A, M) & & \mathbf{C}^0(A, M) &
\end{array}$$

The Bar Resolution

Theorem

If A is a k-algebra and M is an A-bimodule, then

$$\mathrm{H}^n(A, M) = \mathrm{Ext}^n_{A^e}(A, M).$$

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Corollary

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- We have a morphism of k-algebras

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Example

$\varepsilon_i : k[x^2] \to k[x][[t]]$.
Take $\varepsilon_1(x^2) = x^2$, and $\varepsilon_2(x^2) = x^2 + t$.
Not isomorphic as B-algebras.

Secondary Hochschild Cohomology

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- M is an A-bimodule which is B-symmetric.
- Take $\mathbf{C}^n((A, B, \varepsilon); M) = \mathrm{Hom}_k(A^{\otimes n} \otimes B^{\otimes \frac{n(n-1)}{2}}, M)$ in dimension n.

Secondary Hochschild Cohomology

$$0 \longrightarrow M \xrightarrow{\delta_0^{\varepsilon}} \mathrm{Hom}_k(A, M) \xrightarrow{\delta_1^{\varepsilon}} \mathrm{Hom}_k(A^{\otimes 2} \otimes B, M) \xrightarrow{\delta_2^{\varepsilon}}$$

$$\xrightarrow{\delta_2^{\varepsilon}} \mathrm{Hom}_k(A^{\otimes 3} \otimes B^{\otimes 3}, M) \xrightarrow{\delta_3^{\varepsilon}} \cdots$$

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Define the maps $\{\delta_n^\varepsilon\}$ like so:

$$\delta_1^\varepsilon(f)\left(\otimes \begin{pmatrix} a_0 & b_1 \\ 1 & a_1 \end{pmatrix}\right)$$
$$= a_0\varepsilon(b_1)f(a_1) - f(a_0a_1\varepsilon(b_1)) + f(a_0)a_1\varepsilon(b_1).$$

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$$\delta_2^\varepsilon(g)\left(\otimes \begin{pmatrix} a_0 & b_1 & b_2 \\ 1 & a_1 & b_3 \\ 1 & 1 & a_2 \end{pmatrix}\right)$$
$$= a_0\varepsilon(b_1b_2)g\left(\otimes \begin{pmatrix} a_1 & b_3 \\ 1 & a_2 \end{pmatrix}\right) - g\left(\otimes \begin{pmatrix} a_0a_1\varepsilon(b_1) & b_2b_3 \\ 1 & a_2 \end{pmatrix}\right)$$
$$+ g\left(\otimes \begin{pmatrix} a_0 & b_1b_2 \\ 1 & a_1a_2\varepsilon(b_3) \end{pmatrix}\right) - g\left(\otimes \begin{pmatrix} a_0 & b_1 \\ 1 & a_1 \end{pmatrix}\right) a_2\varepsilon(b_2b_3).$$

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Denote the chain as $\mathbf{C}^*((A, B, \varepsilon); M)$.

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Definition

The **secondary Hochschild cohomology** of the triple (A, B, ε) with coefficients in M is denoted as $\mathrm{H}^*((A, B, \varepsilon); M)$ and is defined to be the k-modules of the form

$$\mathrm{H}^n((A, B, \varepsilon); M) = \mathrm{H}^n(\mathbf{C}^*((A, B, \varepsilon); M)) = \frac{\mathrm{Ker}(\delta_n^\varepsilon)}{\mathrm{Im}(\delta_{n-1}^\varepsilon)}.$$

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Theorem

If A is a k-algebra and M is an A-bimodule, then

$$\mathrm{H}^n(A, M) = \mathrm{Ext}^n_{A^e}(A, M).$$

$$\mathrm{H}^n((A, B, \varepsilon); M) = ?$$

Bar-like Resolution

- $\mathbf{Bar}_n(A, B, \varepsilon) = A^{\otimes n+2} \otimes B^{\otimes \frac{(n+1)(n+2)}{2}}$ in dimension n.

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Lemma

$\mathbf{Bar}_*(A, B, \varepsilon)$ *is exact.*

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$$\mathrm{Hom}_{A_1^b}(A^{\otimes 3} \otimes B^{\otimes 3}, M) = \mathrm{Hom}_k(A, M) \qquad \mathrm{Hom}_{A_0^b}(A^{\otimes 2} \otimes B, M) = \mathrm{Hom}_k(k, M)$$

Bar-like Resolution

$$\cdots \xrightarrow{b_2} A^{\otimes 3} \otimes B^{\otimes 3} \xrightarrow{b_1} A^{\otimes 2} \otimes B \xrightarrow{b_0} A \longrightarrow 0.$$

Applying $\mathrm{Hom}_{A^b_n}(-, M)$ we get:

$$\cdots \xleftarrow{b_2^*} \mathrm{Hom}_{A^b_1}(A^{\otimes 3} \otimes B^{\otimes 3}, M) \xleftarrow{b_1^*} \mathrm{Hom}_{A^b_0}(A^{\otimes 2} \otimes B, M) \leftarrow 0.$$

$$\mathrm{Hom}_{A^b_1}(A^{\otimes 3} \otimes B^{\otimes 3}, M) = \mathrm{Hom}_k(A, M) = \mathbf{C}^1((A, B, \varepsilon); M)$$

$$\mathrm{Hom}_{A^b_0}(A^{\otimes 2} \otimes B, M) = \mathrm{Hom}_k(k, M) = \mathbf{C}^0((A, B, \varepsilon); M)$$

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Thank You!