Properties of the secondary Hochschild homology

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January 12, 2018 – JMM, San Diego, CA

Hochschild homology

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- $\otimes = \otimes_k$

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Consider the chain

$$\ldots \longrightarrow M \otimes A^{\otimes n} \xrightarrow{d_n} M \otimes A^{\otimes n-1} \longrightarrow \ldots$$

$$\ldots \longrightarrow M \otimes A^{\otimes 3} \xrightarrow{d_3} M \otimes A^{\otimes 2} \xrightarrow{d_2} M \otimes A \xrightarrow{d_1} M \longrightarrow 0.$$

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Definition (Hochschild – 1945)

The homology of the chain complex above is called the **Hochschild homology of A with coefficients in M**. It is denoted by $\mathrm{H}_\bullet(A, M)$.

Hochschild homology

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Example

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Theorem (Folklore)

For a commutative k-algebra A and an A-symmetric A-bimodule M, we have that

$$\mathrm{H}_1(A, M) \cong M \otimes_A \Omega^1_{A|k}.$$

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$$\dots \longrightarrow M \otimes A^{\otimes n} \otimes B^{\otimes \frac{n(n-1)}{2}} \xrightarrow{\partial_n^{\varepsilon}} M \otimes A^{\otimes n-1} \otimes B^{\otimes \frac{(n-1)(n-2)}{2}} \longrightarrow \dots$$

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Definition (L, Staic, Stancu – 2016)

The homology of the chain complex above is called the **secondary Hochschild homology of the triple** (A, B, ε) **with coefficients in** M. It is denoted by $\mathrm{H}_{\bullet}((A, B, \varepsilon); M)$.

Secondary Hochschild homology

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$$\partial_2^{\varepsilon}\left(m \otimes \begin{pmatrix} a & \alpha \\ 1 & b \end{pmatrix}\right) = ma\varepsilon(\alpha) \otimes b - m \otimes ab\varepsilon(\alpha) + bm\varepsilon(\alpha) \otimes a.$$

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Theorem (L – 2017)

For a commutative triple (A, B, ε) and an A-symmetric A-bimodule M, we have that

$$\mathrm{H}_1((A, B, \varepsilon); M) \cong M \otimes_A \Omega^1_{A|B}.$$

Motivation

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Used Hochschild cohomology to study deformations of algebras that admit a k-algebra structure.

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Used Hochschild homology to pass to cyclic homology.

Hochschild, Kostant, Rosenberg – 1962

Described an isomorphism between $\mathrm{H}_\bullet(A, A)$ and $\Omega^\bullet_{A|k}$.

Exact sequence

Theorem (L – 2017)

Let M be A-symmetric. Then the sequence

$$\mathrm{H}_2(A, M) \xrightarrow{\Phi^2} \mathrm{H}_2((A, B, \varepsilon); M) \xrightarrow{\Psi} \mathrm{H}_1(B, M) \xrightarrow{\varepsilon_*} \mathrm{H}_1(A, M)$$

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Computations

Using the exact sequence, one can easily get:

- $H_1((k,B,\varepsilon);M) = 0$
- $H_2((k,B,\varepsilon);M) \cong H_1(B,M)$
- $H_1((A,A,\mathrm{id});M) = H_2((A,A,\mathrm{id});M) = 0$

Exact sequence

Corollary (First Fundamental Exact Sequence for Ω)

Let $k \longrightarrow B \longrightarrow A$ be morphisms of commutative algebras. Then there is an exact sequence of A-modules

$$A \otimes_B \Omega^1_{B|k} \longrightarrow \Omega^1_{A|k} \longrightarrow \Omega^1_{A|B} \longrightarrow 0.$$

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Proof.

Take A commutative and $M = A$. Apply our exact sequence to get

$$\mathrm{H}_1(B, A) \longrightarrow \mathrm{H}_1(A, A) \longrightarrow \mathrm{H}_1((A, B, \varepsilon); A) \longrightarrow 0.$$

□

Questions

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Thank you!