The Derived Category

Jacob Laubacher

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The derived category $\mathcal{D}(R - \mathbf{mod})$.

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Theorem

$$Ext_R^n(A,B) \cong Hom_{\mathcal{D}(R-\mathbf{mod})}(\mathbf{A}[0],\mathbf{B}[n]).$$

Modules

Let R be a ring with identity.

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Definition

We call M a *left module* of R where $(M, +)$ is an abelian group along with the operation $\cdot : R \times M \to M$ with $(r, m) \mapsto r \cdot m$ such that for all $r, s \in R$ and for all $m, n \in M$ we have that

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(i) $r \cdot (m+n) = r \cdot m + r \cdot n$,

(ii) $(r+s) \cdot m = r \cdot m + s \cdot m$,

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(iii) $r \cdot (s \cdot m) = (rs) \cdot m$, and

(iv) $1_R \cdot m = m$.

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$\mathbb{Z}_n$ and $\mathbb{Q}$ are $\mathbb{Z}$-modules. The operation $\cdot$ is given by multiplication.

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Example

$\mathbb{Z}_n$ and $\mathbb{Q}$ are $\mathbb{Z}$-modules. The operation $\cdot$ is given by multiplication.
$\mathbb{C}$ is a $\mathbb{C}[x, y]$-module with $\cdot$ given by evaluation.

Modules

The category of left R-modules, denoted $R - \mathbf{mod}$.

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(i) $f(m_1 + m_2) = f(m_1) + f(m_2)$, and

(ii) $f(r \cdot m) = r \cdot f(m)$.

Modules

Definition

A chain of modules

$$\cdots \xrightarrow{d_{n+2}} C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \xrightarrow{d_{n-2}} \cdots$$

is a *complex* if $Im(d_{n+1}) \subseteq Ker(d_n)$ for every n.

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The n-th *homology module* of a complex $\mathbf{C}$ is $H_n(\mathbf{C}) = \dfrac{Ker(d_n)}{Im(d_{n+1})}$.

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Definition

A chain complex is *exact* at C_n if $Im(d_{n+1}) = Ker(d_n)$. So $H_n(\mathbf{C}) = 0$.

Modules

Example

The chain complex

$$\cdots \xrightarrow{2} \mathbb{Z}_4 \xrightarrow{2} \mathbb{Z}_4 \xrightarrow{2} \mathbb{Z}_4 \xrightarrow{2} \mathbb{Z}_4 \xrightarrow{2} \cdots$$

is exact.

Modules

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Example

The chain complex

$$\cdots \xrightarrow{4} \mathbb{Z}_8 \xrightarrow{4} \mathbb{Z}_8 \xrightarrow{4} \mathbb{Z}_8 \xrightarrow{4} \mathbb{Z}_8 \xrightarrow{4} \cdots$$

is not exact in any dimension.

Modules

Definition

A module P is called *projective* if for any modules B and C with a surjective module homomorphism $g : B \to C$ and any module homomorphism $\gamma : P \to C$,

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow{\scriptstyle \gamma} & & \\ B & \xrightarrow{g} & C & \longrightarrow & 0. \end{array}$$

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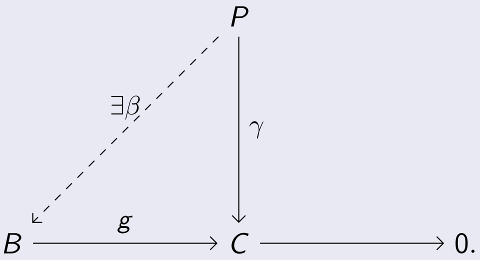

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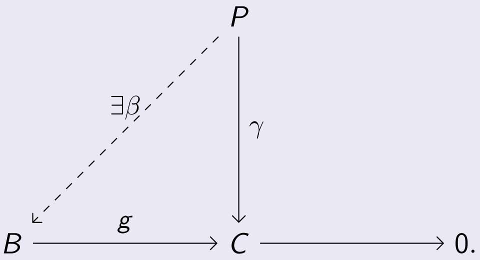

there exists $\beta : P \to B$ such that $\gamma = g \circ \beta$.

Modules

Definition

We define a *projective resolution* of an R-module M as a chain complex $\mathbf{P}$, together with a surjective module homomorphism $\varepsilon : P_0 \to M$ such that

$$\cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

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Theorem

Every R-module has a projective resolution.

Modules

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- Find the cohomology of the resulting chain complex. So

$$Ext^n(A, B) = H^n(Hom(\mathbf{P}, B)) = \frac{Ker(d_{n+1}^*)}{Im(d_n^*)}.$$

Modules

Example

Take $R = \mathbb{Z}$, and $A = \mathbb{Z}_p$. Let B be any $\mathbb{Z}$-module. We want to compute $Ext^n_{\mathbb{Z}}(\mathbb{Z}_p, B)$ for $n \geq 0$.

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and

$$Ext^n_{\mathbb{Z}}(\mathbb{Z}_p, B) = 0 \text{ for all } n \geq 2.$$

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 & & \downarrow{f_{n+1}} & & \downarrow{f_n} & & \downarrow{f_{n-1}} & & \downarrow{f_{n-2}} & & \\
\cdots & \xrightarrow{d_{n+2}} & D_{n+1} & \xrightarrow{d_{n+1}} & D_n & \xrightarrow{d_n} & D_{n-1} & \xrightarrow{d_{n-1}} & D_{n-2} & \xrightarrow{d_{n-2}} & \cdots
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\end{array}
$$

Definition

A chain map $f : \mathbf{C} \to \mathbf{D}$ is called a *quasi-isomorphism* if the induced maps

$$f_* : H_n(\mathbf{C}) \to H_n(\mathbf{D})$$

are isomorphisms for all n.

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Consider the two chain complexes

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and

$$0 \longrightarrow \mathbb{C}[x,y] \xrightarrow{0} \mathbb{C} \longrightarrow 0.$$

Quasi-isomorphisms

Theorem

Let A be any R-module. For any projective resolution $\mathbf{P} \to A$ given as

$$\cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \longrightarrow 0,$$

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$$\mathbf{P} \qquad \cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow 0 \longrightarrow \cdots$$

and

$$\mathbf{A} \qquad \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow A \longrightarrow 0 \longrightarrow \cdots$$

are quasi-isomorphic.

Quasi-isomorphisms

Proof.

$$\mathbf{P} \qquad \cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{0} 0 \xrightarrow{0} \cdots$$

$$\mathbf{A} \qquad \cdots \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} A \xrightarrow{0} 0 \xrightarrow{0} \cdots$$

Quasi-isomorphisms

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$$
\begin{array}{ccccccccccccc}
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- Each square commutes.

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- Need to show $\varepsilon_* : H_0(\mathbf{P}) \to H_0(\mathbf{A})$ is an isomorphism.

Proof.

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Proof.

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- $H_0(\mathbf{P}) = \dfrac{P_0}{Im(d_1)}$ and $H_0(\mathbf{A}) = A$.
- Consider

$$P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A.$$

Quasi-isomorphisms

Proof.

- $H_0(\mathbf{P}) = \dfrac{P_0}{Im(d_1)}$ and $H_0(\mathbf{A}) = A$.
- Consider

$$P_1 \xrightarrow{\ d_1\ } P_0 \xrightarrow{\ \varepsilon\ } A.$$

- $\dfrac{P_0}{Im(d_1)} = \dfrac{P_0}{Ker(\varepsilon)}$

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Proof.

- $H_0(\mathbf{P}) = \dfrac{P_0}{Im(d_1)}$ and $H_0(\mathbf{A}) = A$.
- Consider

$$P_1 \xrightarrow{\quad d_1 \quad} P_0 \xrightarrow{\quad \varepsilon \quad} A.$$

- $\dfrac{P_0}{Im(d_1)} = \dfrac{P_0}{Ker(\varepsilon)} \cong Im(\varepsilon)$

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- $\dfrac{P_0}{Im(d_1)} = \dfrac{P_0}{Ker(\varepsilon)} \cong Im(\varepsilon) = A.$

□

Localization

Definition

Given a category $\mathcal{C}$ and a collection of morphisms S in $\mathcal{C}$, the *localization* of $\mathcal{C}$ *with respect to* S is a category $S^{-1}\mathcal{C}$ together with a functor

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$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{loc} & S^{-1}\mathcal{C} \\ {\scriptstyle F}\big\downarrow & & \\ \mathcal{A} & & \end{array}$$

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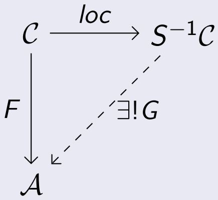

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Example

For any category $\mathcal{C}$, if $S = \{id\}$, then $S^{-1}\mathcal{C} = \mathcal{C}$.

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Example

Consider $\mathbb{Z}$ and $S = \mathbb{Z} \setminus \{0\}$. Then $S^{-1}\mathbb{Z} = \mathbb{Q}$.

Thank You!