Secondary Hochschild and Cyclic (Co)homologies

Jacob Laubacher

Bowling Green State University

February 27, 2017

Defense of Dissertation

Outline of Dissertation

- Simplicial Structures.

Outline of Dissertation

- Simplicial Structures.
- The bar simplicial module $\mathcal{B}(A, B, \varepsilon)$.

Outline of Dissertation

- Simplicial Structures.
- The bar simplicial module $\mathcal{B}(A, B, \varepsilon)$.
- Examples for the secondary and higher order Hochschild (co)homology.

Outline of Dissertation

- Simplicial Structures.
- The bar simplicial module $\mathcal{B}(A, B, \varepsilon)$.
- Examples for the secondary and higher order Hochschild (co)homology.
- Foundations of the secondary cyclic homology.

Outline of Dissertation

- Simplicial Structures.
- The bar simplicial module $\mathcal{B}(A, B, \varepsilon)$.
- Examples for the secondary and higher order Hochschild (co)homology.
- Foundations of the secondary cyclic homology.
- Joint work with Mihai Staic and Alin Stancu.

Goal of the Talk

Theorem

$$\mathrm{H}^\bullet(\mathrm{Hom}_{\mathcal{A}(A\otimes A^{op})}(\mathcal{B}(A),\mathcal{N}(M))) \cong \mathrm{H}^\bullet(A,M).$$

Goal of the Talk

Theorem

$$\mathrm{H}^\bullet(\mathrm{Hom}_{\mathcal{A}(A\otimes A^{op})}(\mathcal{B}(A),\mathcal{N}(M)))) \cong \mathrm{H}^\bullet(A,M).$$

Theorem

$$\mathrm{H}^\bullet(\mathrm{Hom}_{\mathcal{A}(A,B,\varepsilon)}(\mathcal{B}(A,B,\varepsilon),\mathcal{C}(M))) \cong \mathrm{H}^\bullet((A,B,\varepsilon);M).$$

Goal of the Talk

Theorem

$$\mathrm{H}^\bullet(\mathrm{Hom}_{\mathcal{A}(A\otimes A^{op})}(\mathcal{B}(A),\mathcal{N}(M)))) \cong \mathrm{H}^\bullet(A,M).$$

Theorem

$$\mathrm{H}^\bullet(\mathrm{Hom}_{\mathcal{A}(A,B,\varepsilon)}(\mathcal{B}(A,B,\varepsilon),\mathcal{C}(M))) \cong \mathrm{H}^\bullet((A,B,\varepsilon);M).$$

Theorem

$$\mathrm{H}_\bullet(\mathcal{M}^2(M)\otimes_{\mathcal{A}^2(A)}\mathcal{B}^2(A)) \cong \mathrm{H}_\bullet^{S^2}(A,M).$$

Hochschild Cohomology

- Fix k to be a field, $\otimes = \otimes_k$, and all k-algebras have unit.

Hochschild Cohomology

- Fix k to be a field, $\otimes = \otimes_k$, and all k-algebras have unit.
- Let A be an associative k-algebra and M an A-bimodule.

Hochschild Cohomology

- Fix k to be a field, $\otimes = \otimes_k$, and all k-algebras have unit.
- Let A be an associative k-algebra and M an A-bimodule.
- Let B be a commutative k-algebra and $\varepsilon : B \longrightarrow A$ be a morphism of k-algebras such that $\varepsilon(B) \subseteq \mathcal{Z}(A)$ which induces a B-symmetric structure on M.

Hochschild Cohomology

- Fix k to be a field, $\otimes = \otimes_k$, and all k-algebras have unit.
- Let A be an associative k-algebra and M an A-bimodule.
- Let B be a commutative k-algebra and $\varepsilon : B \longrightarrow A$ be a morphism of k-algebras such that $\varepsilon(B) \subseteq \mathcal{Z}(A)$ which induces a B-symmetric structure on M.

Definition (Hochschild - 1945)

The **Hochschild cohomology of A with coefficients in M** is denoted as $\mathrm{H}^\bullet(A, M)$ and is defined to be the cohomology of the cochain complex

$$0 \longrightarrow M \xrightarrow{d^0} \mathrm{Hom}_k(A, M) \xrightarrow{d^1} \mathrm{Hom}_k(A^{\otimes 2}, M) \xrightarrow{d^2} \mathrm{Hom}_k(A^{\otimes 3}, M)$$

$$\xrightarrow{d^3} \dots \xrightarrow{d^{n-1}} \mathrm{Hom}_k(A^{\otimes n}, M) \xrightarrow{d^n} \mathrm{Hom}_k(A^{\otimes n+1}, M) \xrightarrow{d^{n+1}} \dots$$

Hochschild Cohomology

$$0 \longrightarrow M \xrightarrow{d^0} \mathrm{Hom}_k(A, M) \xrightarrow{d^1} \mathrm{Hom}_k(A^{\otimes 2}, M) \xrightarrow{d^2} \mathrm{Hom}_k(A^{\otimes 3}, M)$$

$$\xrightarrow{d^3} \dots \xrightarrow{d^{n-1}} \mathrm{Hom}_k(A^{\otimes n}, M) \xrightarrow{d^n} \mathrm{Hom}_k(A^{\otimes n+1}, M) \xrightarrow{d^{n+1}} \dots$$

Example

$$d^1(f)(a \otimes b) = af(b) - f(ab) + f(a)b.$$

Hochschild Cohomology

$$0 \longrightarrow M \xrightarrow{d^0} \mathrm{Hom}_k(A, M) \xrightarrow{d^1} \mathrm{Hom}_k(A^{\otimes 2}, M) \xrightarrow{d^2} \mathrm{Hom}_k(A^{\otimes 3}, M)$$

$$\xrightarrow{d^3} \dots \xrightarrow{d^{n-1}} \mathrm{Hom}_k(A^{\otimes n}, M) \xrightarrow{d^n} \mathrm{Hom}_k(A^{\otimes n+1}, M) \xrightarrow{d^{n+1}} \dots$$

Example

$$d^1(f)(a \otimes b) = af(b) - f(ab) + f(a)b.$$

Example

$$d^2(g)(a \otimes b \otimes c) = ag(b \otimes c) - g(ab \otimes c) + g(a \otimes bc) - g(a \otimes b)c.$$

Secondary Hochschild Cohomology

Definition (Staic - 2013)

The **secondary Hochschild cohomology of the triple** (A, B, ε) **with coefficients in** M is denoted as $H^\bullet((A, B, \varepsilon); M)$ and is defined to be the cohomology of the cochain complex

$$0 \longrightarrow M \xrightarrow{\delta^0} \mathrm{Hom}_k(A, M) \xrightarrow{\delta^1} \mathrm{Hom}_k(A^{\otimes 2} \otimes B, M) \xrightarrow{\delta^2}$$

$$\mathrm{Hom}_k(A^{\otimes 3} \otimes B^{\otimes 3}, M) \xrightarrow{\delta^3} \ldots \xrightarrow{\delta^{n-1}} \mathrm{Hom}_k(A^{\otimes n} \otimes B^{\otimes \frac{n(n-1)}{2}}, M) \xrightarrow{\delta^n} \ldots$$

Definition (Staic - 2013)

The **secondary Hochschild cohomology of the triple** (A, B, ε) **with coefficients in** M is denoted as $\mathrm{H}^{\bullet}((A, B, \varepsilon); M)$ and is defined to be the cohomology of the cochain complex

$$0 \longrightarrow M \xrightarrow{\delta^0} \mathrm{Hom}_k(A, M) \xrightarrow{\delta^1} \mathrm{Hom}_k(A^{\otimes 2} \otimes B, M) \xrightarrow{\delta^2}$$

$$\mathrm{Hom}_k(A^{\otimes 3} \otimes B^{\otimes 3}, M) \xrightarrow{\delta^3} \ldots \xrightarrow{\delta^{n-1}} \mathrm{Hom}_k(A^{\otimes n} \otimes B^{\otimes \frac{n(n-1)}{2}}, M) \xrightarrow{\delta^n} \ldots$$

Example

$$\delta^1(f)\left(\otimes \begin{pmatrix} a & \alpha \\ 1 & b \end{pmatrix}\right) = a\varepsilon(\alpha)f(b) - f(a\varepsilon(\alpha)b) + f(a)b\varepsilon(\alpha).$$

Secondary Hochschild Cohomology

$$0 \longrightarrow M \xrightarrow{\delta^0} \mathrm{Hom}_k(A, M) \xrightarrow{\delta^1} \mathrm{Hom}_k(A^{\otimes 2} \otimes B, M) \xrightarrow{\delta^2}$$

$$\mathrm{Hom}_k(A^{\otimes 3} \otimes B^{\otimes 3}, M) \xrightarrow{\delta^3} \ldots \xrightarrow{\delta^{n-1}} \mathrm{Hom}_k(A^{\otimes n} \otimes B^{\otimes \frac{n(n-1)}{2}}, M) \xrightarrow{\delta^n} \ldots$$

Example

$$\delta^2(g)\left(\otimes\begin{pmatrix} a & \alpha & \beta \\ 1 & b & \gamma \\ 1 & 1 & c \end{pmatrix}\right)$$

$$= a\varepsilon(\alpha\beta)g\left(\otimes\begin{pmatrix} b & \gamma \\ 1 & c \end{pmatrix}\right) - g\left(\otimes\begin{pmatrix} a\varepsilon(\alpha)b & \beta\gamma \\ 1 & c \end{pmatrix}\right)$$

$$+ g\left(\otimes\begin{pmatrix} a & \alpha\beta \\ 1 & b\varepsilon(\gamma)c \end{pmatrix}\right) - g\left(\otimes\begin{pmatrix} a & \alpha \\ 1 & b \end{pmatrix}\right)\varepsilon(\beta\gamma)c.$$

Higher Order Hochschild Homology

Let A be a commutative k-algebra and M an A-bimodule which is A-symmetric.

Higher Order Hochschild Homology

Let A be a commutative k-algebra and M an A-bimodule which is A-symmetric.

Definition (Anderson - 1971, Pirashvili - 2000)

The **higher order Hochschild homology of** A **with values in** M **over the simplicial set** S^2 is denoted as $\mathrm{H}_\bullet^{S^2}(A,M)$ and is defined to be the homology of the chain complex

$$\dots \xrightarrow{\partial_{n+1}} M \otimes A^{\otimes \frac{n(n-1)}{2}} \xrightarrow{\partial_n} M \otimes A^{\otimes \frac{(n-1)(n-2)}{2}} \xrightarrow{\partial_{n-1}} \dots$$

$$\dots \xrightarrow{\partial_5} M \otimes A^{\otimes 6} \xrightarrow{\partial_4} M \otimes A^{\otimes 3} \xrightarrow{\partial_3} M \otimes A \xrightarrow{\partial_2} M \xrightarrow{\partial_1} M \longrightarrow 0.$$

Higher Order Hochschild Homology

Let A be a commutative k-algebra and M an A-bimodule which is A-symmetric.

Definition (Anderson - 1971, Pirashvili - 2000)

The **higher order Hochschild homology of** A **with values in** M **over the simplicial set** S^2 is denoted as $\mathrm{H}_\bullet^{S^2}(A, M)$ and is defined to be the homology of the chain complex

$$\dots \xrightarrow{\partial_{n+1}} M \otimes A^{\otimes \frac{n(n-1)}{2}} \xrightarrow{\partial_n} M \otimes A^{\otimes \frac{(n-1)(n-2)}{2}} \xrightarrow{\partial_{n-1}} \dots$$

$$\dots \xrightarrow{\partial_5} M \otimes A^{\otimes 6} \xrightarrow{\partial_4} M \otimes A^{\otimes 3} \xrightarrow{\partial_3} M \otimes A \xrightarrow{\partial_2} M \xrightarrow{\partial_1} M \longrightarrow 0.$$

Example

$$\partial_1(m) = m - m = 0.$$

$$\partial_2(m \otimes a) = ma - ma + ma = ma.$$

Higher Order Hochschild Homology

$$\ldots \xrightarrow{\partial_5} M \otimes A^{\otimes 6} \xrightarrow{\partial_4} M \otimes A^{\otimes 3} \xrightarrow{\partial_3} M \otimes A \xrightarrow{\partial_2} M \xrightarrow{\partial_1} M \longrightarrow 0.$$

Example

$$\partial_3 \left(m \otimes \begin{pmatrix} a & b \\ 1 & c \end{pmatrix} \right) = mab \otimes c - ma \otimes bc + mc \otimes ab - mbc \otimes a.$$

$$\dots \xrightarrow{\partial_5} M \otimes A^{\otimes 6} \xrightarrow{\partial_4} M \otimes A^{\otimes 3} \xrightarrow{\partial_3} M \otimes A \xrightarrow{\partial_2} M \xrightarrow{\partial_1} M \longrightarrow 0.$$

Example

$$\partial_3 \left(m \otimes \begin{pmatrix} a & b \\ 1 & c \end{pmatrix} \right) = mab \otimes c - ma \otimes bc + mc \otimes ab - mbc \otimes a.$$

Example

$$\partial_4 \left(m \otimes \begin{pmatrix} a & b & c \\ 1 & d & e \\ 1 & 1 & f \end{pmatrix} \right) = mabc \otimes \begin{pmatrix} d & e \\ 1 & f \end{pmatrix} - ma \otimes \begin{pmatrix} bd & ce \\ 1 & f \end{pmatrix}$$

$$+ md \otimes \begin{pmatrix} ab & c \\ 1 & ef \end{pmatrix} - mf \otimes \begin{pmatrix} a & bc \\ 1 & de \end{pmatrix} + mcef \otimes \begin{pmatrix} a & b \\ 1 & d \end{pmatrix}.$$

Motivation

Remark (Hochschild - 1945)

$\mathrm{H}^\bullet(A, M)$ can be used to study extensions of algebras over a field.

Motivation

Remark (Hochschild - 1945)

$\mathrm{H}^\bullet(A, M)$ can be used to study extensions of algebras over a field.

Theorem (Gerstenhaber - 1964)

$\mathrm{H}^\bullet(A, M)$ can be used to study deformations of algebras. The multiplication law

$$m_t(a \otimes b) = ab + c_1(a \otimes b)t + c_2(a \otimes b)t^2 + c_3(a \otimes b)t^3 + \cdots$$

is associative mod t^2 if and only if c_1 is a 2-cocycle.

Motivation

Remark (Hochschild - 1945)

$\mathrm{H}^\bullet(A, M)$ can be used to study extensions of algebras over a field.

Theorem (Gerstenhaber - 1964)

$\mathrm{H}^\bullet(A, M)$ can be used to study deformations of algebras. The multiplication law

$$m_t(a \otimes b) = ab + c_1(a \otimes b)t + c_2(a \otimes b)t^2 + c_3(a \otimes b)t^3 + \cdots$$

is associative mod t^2 if and only if c_1 is a 2-cocycle.

Remark (Staic - 2013)

The secondary Hochschild cohomology $\mathrm{H}^\bullet((A, B, \varepsilon); M)$ can be used to study deformations of B-algebras.

Motivation

Remark (Hochschild - 1945)

$\mathrm{H}^\bullet(A, M)$ can be used to study extensions of algebras over a field.

Theorem (Gerstenhaber - 1964)

$\mathrm{H}^\bullet(A, M)$ can be used to study deformations of algebras. The multiplication law

$$m_t(a \otimes b) = ab + c_1(a \otimes b)t + c_2(a \otimes b)t^2 + c_3(a \otimes b)t^3 + \cdots$$

is associative mod t^2 if and only if c_1 is a 2-cocycle.

Remark (Staic - 2013)

The secondary Hochschild cohomology $\mathrm{H}^\bullet((A, B, \varepsilon); M)$ can be used to study deformations of B-algebras.

Remark (Ginot, Tradler, and Zeinalian - 2014)

Higher order (co)homology applies to brane and string topology.

Goal of the Talk

Theorem

$$H^\bullet(\mathrm{Hom}_{\mathcal{A}(A\otimes A^{op})}(\mathcal{B}(A),\mathcal{N}(M))) \cong H^\bullet(A,M).$$

$$H^\bullet(\mathrm{Hom}_{\mathcal{A}(A,B,\varepsilon)}(\mathcal{B}(A,B,\varepsilon),\mathcal{C}(M))) \cong H^\bullet((A,B,\varepsilon);M).$$

$$H_\bullet(\mathcal{M}^2(M)\otimes_{\mathcal{A}^2(A)}\mathcal{B}^2(A)) \cong H_\bullet^{S^2}(A,M).$$

Goal of the Talk

Theorem

$$\mathrm{H}^\bullet(\mathrm{Hom}_{\mathcal{A}(A\otimes A^{op})}(\mathcal{B}(A),\mathcal{N}(M))) \cong \mathrm{H}^\bullet(A,M).$$

$$\mathrm{H}^\bullet(\mathrm{Hom}_{\mathcal{A}(A,B,\varepsilon)}(\mathcal{B}(A,B,\varepsilon),\mathcal{C}(M))) \cong \mathrm{H}^\bullet((A,B,\varepsilon);M).$$

$$\mathrm{H}_\bullet(\mathcal{M}^2(M)\otimes_{\mathcal{A}^2(A)}\mathcal{B}^2(A)) \cong \mathrm{H}_\bullet^{S^2}(A,M).$$

We'll study the simplicial structure of the complexes associated to:

Goal of the Talk

Theorem

$$\mathrm{H}^{\bullet}(\mathrm{Hom}_{\mathcal{A}(A\otimes A^{op})}(\mathcal{B}(A),\mathcal{N}(M))) \cong \mathrm{H}^{\bullet}(A,M).$$

$$\mathrm{H}^{\bullet}(\mathrm{Hom}_{\mathcal{A}(A,B,\varepsilon)}(\mathcal{B}(A,B,\varepsilon),\mathcal{C}(M))) \cong \mathrm{H}^{\bullet}((A,B,\varepsilon);M).$$

$$\mathrm{H}_{\bullet}(\mathcal{M}^2(M)\otimes_{\mathcal{A}^2(A)}\mathcal{B}^2(A)) \cong \mathrm{H}_{\bullet}^{S^2}(A,M).$$

We'll study the simplicial structure of the complexes associated to:

- the Hochschild cohomology of A with coefficients in M, $\mathrm{H}^{\bullet}(A,M)$.

Goal of the Talk

Theorem

$$\mathrm{H}^\bullet(\mathrm{Hom}_{\mathcal{A}(A\otimes A^{op})}(\mathcal{B}(A),\mathcal{N}(M)))\cong \mathrm{H}^\bullet(A,M).$$

$$\mathrm{H}^\bullet(\mathrm{Hom}_{\mathcal{A}(A,B,\varepsilon)}(\mathcal{B}(A,B,\varepsilon),\mathcal{C}(M)))\cong \mathrm{H}^\bullet((A,B,\varepsilon);M).$$

$$\mathrm{H}_\bullet(\mathcal{M}^2(M)\otimes_{\mathcal{A}^2(A)}\mathcal{B}^2(A))\cong \mathrm{H}_\bullet^{S^2}(A,M).$$

We'll study the simplicial structure of the complexes associated to:

- the Hochschild cohomology of A with coefficients in M, $\mathrm{H}^\bullet(A,M)$.
- the secondary Hochschild cohomology of the triple (A,B,ε) with coefficients in M, $\mathrm{H}^\bullet((A,B,\varepsilon);M)$.

Goal of the Talk

Theorem

$$\mathrm{H}^\bullet(\mathrm{Hom}_{\mathcal{A}(A\otimes A^{op})}(\mathcal{B}(A),\mathcal{N}(M)))\cong \mathrm{H}^\bullet(A,M).$$

$$\mathrm{H}^\bullet(\mathrm{Hom}_{\mathcal{A}(A,B,\varepsilon)}(\mathcal{B}(A,B,\varepsilon),\mathcal{C}(M)))\cong \mathrm{H}^\bullet((A,B,\varepsilon);M).$$

$$\mathrm{H}_\bullet(\mathcal{M}^2(M)\otimes_{\mathcal{A}^2(A)}\mathcal{B}^2(A))\cong \mathrm{H}_\bullet^{S^2}(A,M).$$

We'll study the simplicial structure of the complexes associated to:

- the Hochschild cohomology of A with coefficients in M, $\mathrm{H}^\bullet(A,M)$.
- the secondary Hochschild cohomology of the triple (A,B,ε) with coefficients in M, $\mathrm{H}^\bullet((A,B,\varepsilon);M)$.
- the higher order Hochschild homology of A with values in M over the simplicial set S^2, $\mathrm{H}_\bullet^{S^2}(A,M)$.

Key Lemmas

Lemma (L, Staic, and Stancu - 2016)

Suppose that $(\mathcal{X}, \delta_i^{\mathcal{X}}, \sigma_i^{\mathcal{X}})$ *is a simplicial left module over the simplicial k-algebra* $\mathcal{A}$ *and* $(\mathcal{Y}, \delta_{\mathcal{Y}}^i, \sigma_{\mathcal{Y}}^i)$ *is a co-simplicial left module over the simplicial k-algebra* $\mathcal{A}$*. Then* $(\mathrm{Hom}_{\mathcal{A}}(\mathcal{X}, \mathcal{Y}), D^i, S^i)$ *is a co-simplicial k-module.*

Key Lemmas

Lemma (L, Staic, and Stancu - 2016)

Suppose that $(\mathcal{X}, \delta_i^{\mathcal{X}}, \sigma_i^{\mathcal{X}})$ *is a simplicial left module over the simplicial k-algebra* $\mathcal{A}$ *and* $(\mathcal{Y}, \delta^i_{\mathcal{Y}}, \sigma^i_{\mathcal{Y}})$ *is a co-simplicial left module over the simplicial k-algebra* $\mathcal{A}$*. Then* $(\mathrm{Hom}_{\mathcal{A}}(\mathcal{X}, \mathcal{Y}), D^i, S^i)$ *is a co-simplicial k-module.*

Lemma (L, Staic, and Stancu - 2016)

Suppose that $(\mathcal{X}, \delta_i^{\mathcal{X}}, \sigma_i^{\mathcal{X}})$ *is a simplicial right module over the simplicial k-algebra* $\mathcal{A}$*, and* $(\mathcal{Y}, \delta_i^{\mathcal{Y}}, \sigma_i^{\mathcal{Y}})$ *is a simplicial left module over the simplicial k-algebra* $\mathcal{A}$*. Then* $(\mathcal{X} \otimes_{\mathcal{A}} \mathcal{Y}, D_i, S_i)$ *is a simplicial k-module.*

Simplicial Identities

Recall the following: For

$$\delta_i : X_n \longrightarrow X_{n-1}$$

and

$$\sigma_i : X_n \longrightarrow X_{n+1},$$

where $0 \leq i \leq n$, we have:

Simplicial Identities

Recall the following: For

$$\delta_i : X_n \longrightarrow X_{n-1}$$

and

$$\sigma_i : X_n \longrightarrow X_{n+1},$$

where $0 \leq i \leq n$, we have:

The Simplicial Identities

$$\begin{aligned}
\delta_i\delta_j &= \delta_{j-1}\delta_i \quad \text{if} \quad i < j,\\
\sigma_i\sigma_j &= \sigma_{j+1}\sigma_i \quad \text{if} \quad i \leq j,\\
\delta_i\sigma_j &= \sigma_{j-1}\delta_i \quad \text{if} \quad i < j,\\
\delta_i\sigma_j &= \text{id} \quad \text{if} \quad i = j \quad \text{or} \quad i = j+1,\\
\delta_i\sigma_j &= \sigma_j\delta_{i-1} \quad \text{if} \quad i > j+1.
\end{aligned}$$

Simplicial Algebras

Definition

A **simplicial k-algebra** is a collection of k-algebras A_n together with morphisms of k-algebras

$$\delta_i^{\mathcal{A}} : A_n \longrightarrow A_{n-1} \quad \text{and} \quad \sigma_i^{\mathcal{A}} : A_n \longrightarrow A_{n+1}$$

such that the simplicial identities are satisfied.

Simplicial Algebras

Definition

A **simplicial k-algebra** is a collection of k-algebras A_n together with morphisms of k-algebras

$$\delta_i^{\mathcal{A}} : A_n \longrightarrow A_{n-1} \quad \text{and} \quad \sigma_i^{\mathcal{A}} : A_n \longrightarrow A_{n+1}$$

such that the simplicial identities are satisfied.

Example

We define the simplicial k-algebra $\mathcal{A}(A \otimes A^{op})$ by setting $A_n = A \otimes A^{op}$, $\delta_i^{\mathcal{A}} = \mathrm{id}_{A \otimes A^{op}}$, and $\sigma_i^{\mathcal{A}} = \mathrm{id}_{A \otimes A^{op}}$.

Simplicial Algebras

Definition

A **simplicial k-algebra** is a collection of k-algebras A_n together with morphisms of k-algebras

$$\delta_i^{\mathcal{A}} : A_n \longrightarrow A_{n-1} \quad \text{and} \quad \sigma_i^{\mathcal{A}} : A_n \longrightarrow A_{n+1}$$

such that the simplicial identities are satisfied.

Example

We define the simplicial k-algebra $\mathcal{A}(A \otimes A^{op})$ by setting $A_n = A \otimes A^{op}$, $\delta_i^{\mathcal{A}} = \mathrm{id}_{A \otimes A^{op}}$, and $\sigma_i^{\mathcal{A}} = \mathrm{id}_{A \otimes A^{op}}$.

Example

We define the simplicial k-algebra $\mathcal{A}(A, B, \varepsilon)$ by setting $A_n = A \otimes B^{\otimes 2n+1} \otimes A^{op}$. $\delta_i^{\mathcal{A}}$ and $\sigma_i^{\mathcal{A}}$ are defined naturally.

Simplicial Algebras

Let A be a commutative k-algebra.

Let A be a commutative k-algebra.

Example

$\mathcal{A}^2(A)$ is a simplicial k-algebra by setting $A_n = A^{\otimes 3n+3}$ for all n.

$$\delta_1^{\mathcal{A}}\left(\otimes\begin{pmatrix} a_0 & b_1 & b_2 & b_3 & d \\ & a_1 & & & c_1 \\ & & a_2 & & c_2 \\ & & & a_3 & c_3 \\ & & & & a_4 \end{pmatrix}\right) = \otimes\begin{pmatrix} a_0 & b_1 b_2 & b_3 & d \\ & a_1 a_2 & & c_1 c_2 \\ & & a_3 & c_3 \\ & & & a_4 \end{pmatrix},$$

and $\sigma_i^{\mathcal{A}}$ is defined similarly.

Simplicial Left Modules

Definition

We say that $\mathcal{M}$ is a **simplicial left module** over a simplicial k-algebra $\mathcal{A}$ if $\mathcal{M} = (M_n)_{n\geq 0}$ is a simplicial k-vector space together with a left A_n-module structure on M_n for all $n \geq 0$ such that we have the following natural compatibility conditions:

$$\delta_i^{\mathcal{M}}(a_n m_n) = \delta_i^{\mathcal{A}}(a_n)\delta_i^{\mathcal{M}}(m_n)$$

and

$$\sigma_i^{\mathcal{M}}(a_n m_n) = \sigma_i^{\mathcal{A}}(a_n)\sigma_i^{\mathcal{M}}(m_n)$$

for all $a_n \in A_n$, for all $m_n \in M_n$, and for $0 \leq i \leq n$.

Simplicial Left Modules

Definition

We say that $\mathcal{M}$ is a **simplicial left module** over a simplicial k-algebra $\mathcal{A}$ if $\mathcal{M} = (M_n)_{n\geq 0}$ is a simplicial k-vector space together with a left A_n-module structure on M_n for all $n \geq 0$ such that we have the following natural compatibility conditions:

$$\delta_i^{\mathcal{M}}(a_n m_n) = \delta_i^{\mathcal{A}}(a_n)\delta_i^{\mathcal{M}}(m_n)$$

and

$$\sigma_i^{\mathcal{M}}(a_n m_n) = \sigma_i^{\mathcal{A}}(a_n)\sigma_i^{\mathcal{M}}(m_n)$$

for all $a_n \in A_n$, for all $m_n \in M_n$, and for $0 \leq i \leq n$.

Remark

We can define **simplicial right modules** in a similar way.

Example

We define the simplicial left module $\mathcal{B}(A)$ over the simplicial k-algebra $\mathcal{A}(A \otimes A^{op})$ by setting $B_n = A^{\otimes n+2}$ for all $n \geq 0$.

Simplicial Left Modules

Example

We define the simplicial left module $\mathcal{B}(A)$ over the simplicial k-algebra $\mathcal{A}(A \otimes A^{op})$ by setting $B_n = A^{\otimes n+2}$ for all $n \geq 0$.

Example

We define the simplicial left module $\mathcal{B}(A, B, \varepsilon)$ over the simplicial k-algebra $\mathcal{A}(A, B, \varepsilon)$ by setting

$$B_n = A^{\otimes n+2} \otimes B^{\otimes \frac{(n+1)(n+2)}{2}}$$

for all $n \geq 0$.
The left module structure, $\delta_i^{\mathcal{B}}$, and $\sigma_i^{\mathcal{B}}$ are all defined naturally.

Simplicial Left Modules

Example

$\mathcal{B}^2(A)$ is a simplicial left module over $\mathcal{A}^2(A)$ by setting $B_n = A^{\otimes \frac{(n+2)(n+3)}{2}}$ for all $n \geq 0$. The left module structure:

$$\otimes \begin{pmatrix} a_0 & b_1 & b_2 & b_3 & d \\ & a_1 & & & c_1 \\ & & a_2 & & c_2 \\ & & & a_3 & c_3 \\ & & & & a_4 \end{pmatrix} \cdot \otimes \begin{pmatrix} a_{0,0} & a_{0,1} & a_{0,2} & a_{0,3} & a_{0,4} \\ 1 & a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ 1 & 1 & a_{2,2} & a_{2,3} & a_{2,4} \\ 1 & 1 & 1 & a_{3,3} & a_{3,4} \\ 1 & 1 & 1 & 1 & a_{4,4} \end{pmatrix}$$

$$= \otimes \begin{pmatrix} a_0 a_{0,0} & b_1 a_{0,1} & b_2 a_{0,2} & b_3 a_{0,3} & d a_{0,4} \\ 1 & a_1 a_{1,1} & a_{1,2} & a_{1,3} & c_1 a_{1,4} \\ 1 & 1 & a_2 a_{2,2} & a_{2,3} & c_2 a_{2,4} \\ 1 & 1 & 1 & a_3 a_{3,3} & c_3 a_{3,4} \\ 1 & 1 & 1 & 1 & a_4 a_{4,4} \end{pmatrix}.$$

Simplicial Right Modules

Let M be an A-bimodule which is A-symmetric.

Simplicial Right Modules

Let M be an A-bimodule which is A-symmetric.

Example

$\mathcal{M}^2(M)$ is a simplicial right module over $\mathcal{A}^2(A)$ by setting $M_n = M$ for all $n \geq 0$. The right module structure is

$$m \cdot \otimes \begin{pmatrix} a_0 & b_1 & b_2 & b_3 & d \\ & a_1 & & & c_1 \\ & & a_2 & & c_2 \\ & & & a_3 & c_3 \\ & & & & a_4 \end{pmatrix} = ma_0a_1a_2a_3a_4b_1b_2b_3c_1c_2c_3d.$$

Again, $\delta_i^{\mathcal{M}} = \mathrm{id}_M$ and $\sigma_i^{\mathcal{M}} = \mathrm{id}_M$.

Co-simplicial Left Modules

Definition

We say that $\mathcal{M}$ is a **co-simplicial left module** over a simplicial k-algebra $\mathcal{A}$ if $\mathcal{M} = (M^n)_{n\geq 0}$ is a co-simplicial k-vector space together with a left A_n-module structure on M^n for all $n \geq 0$ such that we have the following natural compatibility conditions:

$$\delta^i_{\mathcal{M}}(\delta^{\mathcal{A}}_i(a_{n+1}) \cdot m_n) = a_{n+1} \cdot \delta^i_{\mathcal{M}}(m_n)$$

and

$$\sigma^i_{\mathcal{M}}(\sigma^{\mathcal{A}}_i(a_{n-1}) \cdot m_n) = a_{n-1} \cdot \sigma^i_{\mathcal{M}}(m_n)$$

for all $a_{n-1} \in A_{n-1}$, for all $a_{n+1} \in A_{n+1}$, for all $m_n \in M_n$, and for $0 \leq i \leq n$.

Co-simplicial Left Modules

Example

$\mathcal{N}(M)$ is the co-simplicial left module over $\mathcal{A}(A \otimes A^{op})$ obtained by setting $N^n = M$ for all n, $\delta^i_{\mathcal{N}} = \mathrm{id}_M$, and $\sigma^i_{\mathcal{N}} = \mathrm{id}_M$.

Co-simplicial Left Modules

Example

$\mathcal{N}(M)$ is the co-simplicial left module over $\mathcal{A}(A \otimes A^{op})$ obtained by setting $N^n = M$ for all n, $\delta^i_{\mathcal{N}} = \mathrm{id}_M$, and $\sigma^i_{\mathcal{N}} = \mathrm{id}_M$.

Example

$\mathcal{C}(M)$ is the co-simplicial left module over $\mathcal{A}(A, B, \varepsilon)$ obtained by setting $C^n = M$ for all n, $\delta^i_{\mathcal{C}} = \mathrm{id}_M$, and $\sigma^i_{\mathcal{C}} = \mathrm{id}_M$.

Key Lemma

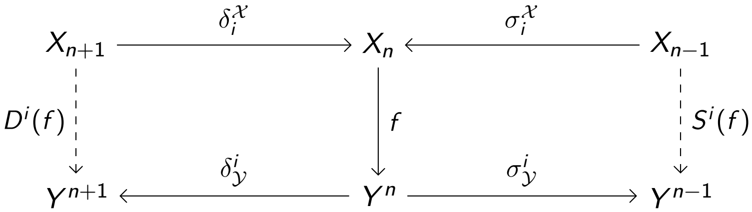
Lemma (L, Staic, and Stancu - 2016)

Suppose that $(\mathcal{X}, \delta_i^{\mathcal{X}}, \sigma_i^{\mathcal{X}})$ *is a simplicial left module over the simplicial k-algebra* $\mathcal{A}$ *and* $(\mathcal{Y}, \delta^i_{\mathcal{Y}}, \sigma^i_{\mathcal{Y}})$ *is a co-simplicial left module over the simplicial k-algebra* $\mathcal{A}$*. Then* $(\mathrm{Hom}_{\mathcal{A}}(\mathcal{X}, \mathcal{Y}), D^i, S^i)$ *is a co-simplicial k-module.*

Key Lemma

Lemma (L, Staic, and Stancu - 2016)

Suppose that $(\mathcal{X}, \delta_i^{\mathcal{X}}, \sigma_i^{\mathcal{X}})$ *is a simplicial left module over the simplicial k-algebra* $\mathcal{A}$ *and* $(\mathcal{Y}, \delta^i_{\mathcal{Y}}, \sigma^i_{\mathcal{Y}})$ *is a co-simplicial left module over the simplicial k-algebra* $\mathcal{A}$*. Then* $(\mathrm{Hom}_{\mathcal{A}}(\mathcal{X}, \mathcal{Y}), D^i, S^i)$ *is a co-simplicial k-module.*



Results

Theorem

$$\mathrm{Hom}_{\mathcal{A}(A\otimes A^{op})}(\mathcal{B}(A),\mathcal{N}(M))$$

is a co-simplicial k-module, and

$$\begin{aligned}\mathrm{H}^\bullet(\mathrm{Hom}_{\mathcal{A}(A\otimes A^{op})}(\mathcal{B}(A),\mathcal{N}(M))) &\cong \mathrm{H}^\bullet(\mathrm{Hom}_{A\otimes A^{op}}(\mathcal{B}(A), M)) \\ &\cong \mathrm{H}^\bullet(A, M).\end{aligned}$$

Theorem

$$\mathrm{Hom}_{\mathcal{A}(A\otimes A^{op})}(\mathcal{B}(A),\mathcal{N}(M))$$

is a co-simplicial k-module, and

$$\mathrm{H}^{\bullet}(\mathrm{Hom}_{\mathcal{A}(A\otimes A^{op})}(\mathcal{B}(A),\mathcal{N}(M))) \cong \mathrm{H}^{\bullet}(\mathrm{Hom}_{A\otimes A^{op}}(\mathcal{B}(A), M)) \cong \mathrm{H}^{\bullet}(A, M).$$

Theorem (L, Staic, and Stancu - 2016)

$$\mathrm{Hom}_{\mathcal{A}(A,B,\varepsilon)}(\mathcal{B}(A,B,\varepsilon),\mathcal{C}(M))$$

is a co-simplicial k-module, and

$$\mathrm{H}^{\bullet}(\mathrm{Hom}_{\mathcal{A}(A,B,\varepsilon)}(\mathcal{B}(A,B,\varepsilon),\mathcal{C}(M))) \cong \mathrm{H}^{\bullet}((A,B,\varepsilon);M).$$

Lemma (L, Staic, and Stancu - 2016)

Suppose that $(\mathcal{X}, \delta_i^{\mathcal{X}}, \sigma_i^{\mathcal{X}})$ *is a simplicial right module over the simplicial k-algebra* $\mathcal{A}$*, and* $(\mathcal{Y}, \delta_i^{\mathcal{Y}}, \sigma_i^{\mathcal{Y}})$ *is a simplicial left module over the simplicial k-algebra* $\mathcal{A}$*. Then* $(\mathcal{X} \otimes_{\mathcal{A}} \mathcal{Y}, D_i, S_i)$ *is a simplicial k-module.*

Results

Lemma (L, Staic, and Stancu - 2016)

Suppose that $(\mathcal{X}, \delta_i^{\mathcal{X}}, \sigma_i^{\mathcal{X}})$ *is a simplicial right module over the simplicial k-algebra* $\mathcal{A}$*, and* $(\mathcal{Y}, \delta_i^{\mathcal{Y}}, \sigma_i^{\mathcal{Y}})$ *is a simplicial left module over the simplicial k-algebra* $\mathcal{A}$*. Then* $(\mathcal{X} \otimes_{\mathcal{A}} \mathcal{Y}, D_i, S_i)$ *is a simplicial k-module.*

Theorem

$$\mathcal{M}^2(M) \otimes_{\mathcal{A}^2(A)} \mathcal{B}^2(A)$$

is a simplicial k-module, and

$$\mathrm{H}_\bullet(\mathcal{M}^2(M) \otimes_{\mathcal{A}^2(A)} \mathcal{B}^2(A)) \cong \mathrm{H}_\bullet^{S^2}(A, M).$$

Results

Note that we have

$$M_n \otimes_{A_n} B_n = M \otimes_{A^{\otimes 3n+3}} A^{\otimes \frac{(n+2)(n+3)}{2}}$$

in dimension n.

Results

Note that we have

$$M_n \otimes_{A_n} B_n = M \otimes_{A^{\otimes 3n+3}} A^{\otimes \frac{(n+2)(n+3)}{2}}$$

in dimension n.
We make identifications:

$$\begin{array}{ccccc}
M \otimes A^{\otimes 6} & \overset{\partial_4}{\dashrightarrow} & M \otimes A^{\otimes 3} & \overset{\partial_3}{\dashrightarrow} & M \otimes A \\
\Big\downarrow \varphi_4^{-1} & & \varphi_3 \Big\uparrow \Big\downarrow \varphi_3^{-1} & & \Big\uparrow \varphi_2 \\
M \otimes_{A^{\otimes 15}} A^{\otimes 21} & \xrightarrow{\sum(-1)^i D_i} & M \otimes_{A^{\otimes 12}} A^{\otimes 15} & \xrightarrow{\sum(-1)^i D_i} & M \otimes_{A^{\otimes 9}} A^{\otimes 10}
\end{array}$$

Results

$$\ldots \xrightarrow{\partial_{n+1}} M \otimes A^{\otimes \frac{n(n-1)}{2}} \xrightarrow{\partial_n} M \otimes A^{\otimes \frac{(n-1)(n-2)}{2}} \xrightarrow{\partial_{n-1}} \ldots$$

$$\ldots \xrightarrow{\partial_5} M \otimes A^{\otimes 6} \xrightarrow{\partial_4} M \otimes A^{\otimes 3} \xrightarrow{\partial_3} M \otimes A \xrightarrow{\partial_2} M \xrightarrow{\partial_1} M \longrightarrow 0.$$

Results

$$\cdots \xrightarrow{\partial_{n+1}} M \otimes A^{\otimes \frac{n(n-1)}{2}} \xrightarrow{\partial_n} M \otimes A^{\otimes \frac{(n-1)(n-2)}{2}} \xrightarrow{\partial_{n-1}} \cdots$$

$$\cdots \xrightarrow{\partial_5} M \otimes A^{\otimes 6} \xrightarrow{\partial_4} M \otimes A^{\otimes 3} \xrightarrow{\partial_3} M \otimes A \xrightarrow{\partial_2} M \xrightarrow{\partial_1} M \longrightarrow 0.$$

Example

$$\partial_1(m) = m - m = 0$$

$$\partial_2(m \otimes a) = ma - ma + ma = ma$$

$$\partial_3 \left(m \otimes \begin{pmatrix} a & b \\ 1 & c \end{pmatrix} \right) = mab \otimes c - ma \otimes bc + mc \otimes ab - mbc \otimes a$$

Results

$$\cdots \xrightarrow{\partial_{n+1}} M \otimes A^{\otimes \frac{n(n-1)}{2}} \xrightarrow{\partial_n} M \otimes A^{\otimes \frac{(n-1)(n-2)}{2}} \xrightarrow{\partial_{n-1}} \cdots$$

$$\cdots \xrightarrow{\partial_5} M \otimes A^{\otimes 6} \xrightarrow{\partial_4} M \otimes A^{\otimes 3} \xrightarrow{\partial_3} M \otimes A \xrightarrow{\partial_2} M \xrightarrow{\partial_1} M \longrightarrow 0.$$

Example

$$\partial_1(m) = m - m = 0$$

$$\partial_2(m \otimes a) = ma - ma + ma = ma$$

$$\partial_3 \left(m \otimes \begin{pmatrix} a & b \\ 1 & c \end{pmatrix} \right) = mab \otimes c - ma \otimes bc + mc \otimes ab - mbc \otimes a$$

Theorem

$$\mathrm{H}_\bullet(\mathcal{M}^2(M) \otimes_{\mathcal{A}^2(A)} \mathcal{B}^2(A)) \cong \mathrm{H}_\bullet^{S^2}(A, M).$$

Other Results

- The secondary bar simplicial module $\mathcal{B}(A, B, \varepsilon)$ has led to:

Other Results

- The secondary bar simplicial module $\mathcal{B}(A, B, \varepsilon)$ has led to:
  - $\mathrm{H}_\bullet((A, B, \varepsilon); M)$.

Other Results

- The secondary bar simplicial module $\mathcal{B}(A, B, \varepsilon)$ has led to:
  - $\mathrm{H}_\bullet((A, B, \varepsilon); M)$.
  - $\mathrm{HH}_\bullet(A, B, \varepsilon)$.

Other Results

- The secondary bar simplicial module $\mathcal{B}(A, B, \varepsilon)$ has led to:
  - $\mathrm{H}_\bullet((A, B, \varepsilon); M)$.
  - $\mathrm{HH}_\bullet(A, B, \varepsilon)$.
  - $\mathrm{HC}_\bullet(A, B, \varepsilon)$.

Other Results

- The secondary bar simplicial module $\mathcal{B}(A, B, \varepsilon)$ has led to:
  - $\mathrm{H}_\bullet((A, B, \varepsilon); M)$.
  - $\mathrm{HH}_\bullet(A, B, \varepsilon)$.
  - $\mathrm{HC}_\bullet(A, B, \varepsilon)$.
- Three equivalent definitions of the secondary cyclic homology associated to (A, B, ε).

Other Results

- The secondary bar simplicial module $\mathcal{B}(A, B, \varepsilon)$ has led to:
  - $\mathrm{H}_\bullet((A, B, \varepsilon); M)$.
  - $\mathrm{HH}_\bullet(A, B, \varepsilon)$.
  - $\mathrm{HC}_\bullet(A, B, \varepsilon)$.
- Three equivalent definitions of the secondary cyclic homology associated to (A, B, ε).
- A secondary analogue of Connes' long exact sequence for both homology and cohomology:

$$\mathrm{HH}_n(A, B, \varepsilon) \xrightarrow{I} \mathrm{HC}_n(A, B, \varepsilon) \xrightarrow{S} \mathrm{HC}_{n-2}(A, B, \varepsilon) \xrightarrow{B} \mathrm{HH}_{n-1}(A, B, \varepsilon)$$

Future Work

Problem

What happens when B is noncommutative?

Future Work

Problem

What happens when B is noncommutative?

Answer

We don't get a complex anymore (need a way around this). This has been an ongoing project with Sam Carolus and Mihai Staic.

Future Work

Problem

What happens when B is noncommutative?

Answer

We don't get a complex anymore (need a way around this). This has been an ongoing project with Sam Carolus and Mihai Staic.

Adequately answering the above would (hopefully) lead to:

Future Work

Problem

What happens when B is noncommutative?

Answer

We don't get a complex anymore (need a way around this). This has been an ongoing project with Sam Carolus and Mihai Staic.

Adequately answering the above would (hopefully) lead to:

- a version of $\mathrm{HH}_\bullet(-)$ preserving Morita equivalence.

Future Work

Problem

What happens when B is noncommutative?

Answer

We don't get a complex anymore (need a way around this). This has been an ongoing project with Sam Carolus and Mihai Staic.

Adequately answering the above would (hopefully) lead to:

- a version of $HH_\bullet(-)$ preserving Morita equivalence.
- a secondary Chern character.

Questions

Thank You!