# The Bar Simplicial Module

joint work with Mihai D. Staic and Alin Stancu

Jacob Laubacher

Bowling Green State University

January 7, 2017

Joint Mathematics Meeting - Atlanta, GA

BGSU®

# Hochschild Cohomology

Fix $k$ to be a field and $\otimes = \otimes_k$. Let $A$ be a $k$-algebra and $M$ an $A$-bimodule.

# Hochschild Cohomology

Fix $k$ to be a field and $\otimes = \otimes_k$. Let $A$ be a $k$-algebra and $M$ an $A$-bimodule.

## Definition (Hochschild - 1945)

**Hochschild cohomology** is $\mathrm{Ext}^n_{A\otimes A^{op}}(A, M)$, which can be shown using the bar resolution $\mathcal{B}(A)$. Denoted $\mathrm{H}^\bullet(A, M)$, it is defined to be the cohomology of the resulting cochain complex:

$$0 \longrightarrow M \xrightarrow{d^0} \mathrm{Hom}_k(A, M) \xrightarrow{d^1} \dots \xrightarrow{d^{n-1}} \mathrm{Hom}_k(A^{\otimes n}, M) \xrightarrow{d^n} \dots$$

# Hochschild Cohomology

Fix $k$ to be a field and $\otimes = \otimes_k$. Let $A$ be a $k$-algebra and $M$ an $A$-bimodule.

**Definition (Hochschild - 1945)**

**Hochschild cohomology** is $\mathrm{Ext}^n_{A\otimes A^{op}}(A, M)$, which can be shown using the bar resolution $\mathcal{B}(A)$. Denoted $\mathrm{H}^\bullet(A, M)$, it is defined to be the cohomology of the resulting cochain complex:

$$0 \longrightarrow M \xrightarrow{d^0} \mathrm{Hom}_k(A, M) \xrightarrow{d^1} \dots \xrightarrow{d^{n-1}} \mathrm{Hom}_k(A^{\otimes n}, M) \xrightarrow{d^n} \dots$$

**Remark (Gerstenhaber - 1964)**

The multiplication law

$$m_t(a \otimes b) = ab + c_1(a \otimes b)t + c_2(a \otimes b)t^2 + c_3(a \otimes b)t^3 + \cdots$$

is associative *mod* $t^2$ if and only if $c_1$ is a 2-cocycle.
The obstruction to extend associativity is in $\mathrm{H}^3(A, A)$.

Let $A$ an associative $k$-algebra, $B$ a commutative $k$-algebra, $\varepsilon : B \longrightarrow A$ a morphism of $k$-algebras such that $\varepsilon(B) \subseteq Z(A)$, and $M$ an $A$-bimodule which is $B$-symmetric.

# Secondary Hochschild Cohomology

Let $A$ an associative $k$-algebra, $B$ a commutative $k$-algebra, $\varepsilon : B \longrightarrow A$ a morphism of $k$-algebras such that $\varepsilon(B) \subseteq Z(A)$, and $M$ an $A$-bimodule which is $B$-symmetric.

**Definition (Staic - 2013)**

The **secondary Hochschild cohomology** of the triple $(A, B, \varepsilon)$ with coefficients in $M$ is denoted as $\mathrm{H}^\bullet((A, B, \varepsilon); M)$ and is defined to be the cohomology of the cochain complex

$$0 \longrightarrow M \xrightarrow{\delta^0} \mathrm{Hom}_k(A, M) \xrightarrow{\delta^1} \mathrm{Hom}_k(A^{\otimes 2} \otimes B, M) \xrightarrow{\delta^2}$$

$$\mathrm{Hom}_k(A^{\otimes 3} \otimes B^{\otimes 3}, M) \xrightarrow{\delta^3} \ldots \xrightarrow{\delta^{n-1}} \mathrm{Hom}_k(A^{\otimes n} \otimes B^{\otimes \frac{n(n-1)}{2}}, M) \xrightarrow{\delta^n} \ldots$$

**Example**

$$d^2(f)(a \otimes b \otimes c) = af(b \otimes c) - f(ab \otimes c) + f(a \otimes bc) - f(a \otimes b)c.$$

**Example**

$$d^2(f)(a\otimes b\otimes c) = af(b\otimes c) - f(ab\otimes c) + f(a\otimes bc) - f(a\otimes b)c.$$

**Example**

$$\begin{aligned}\delta^2(f)\left(\otimes\begin{pmatrix} a & \alpha & \beta \\ 1 & b & \gamma \\ 1 & 1 & c\end{pmatrix}\right) &= a\varepsilon(\alpha\beta)f\left(\otimes\begin{pmatrix} b & \gamma \\ 1 & c\end{pmatrix}\right) - f\left(\otimes\begin{pmatrix} a\varepsilon(\alpha)b & \beta\gamma \\ 1 & c\end{pmatrix}\right) \\ &+ f\left(\otimes\begin{pmatrix} a & \alpha\beta \\ 1 & b\varepsilon(\gamma)c\end{pmatrix}\right) - f\left(\otimes\begin{pmatrix} a & \alpha \\ 1 & b\end{pmatrix}\right)\varepsilon(\beta\gamma)c.\end{aligned}$$

**Theorem (Staic - 2013)**

*Let $A$ be a $k$-algebra, $B$ a commutative $k$-algebra, and $\varepsilon : B \longrightarrow A$ a morphism of $k$-algebras such that $\varepsilon(B) \subseteq Z(A)$. Suppose that for each $i \geq 1$ we have a $k$-linear map $c_i : A \otimes A \otimes B \longrightarrow A$. We consider a family of products $\mathcal{M} = \{m_{\alpha,t}\}_{\alpha \in B}$ where*

$$m_{\alpha,t}(a \otimes b) = \varepsilon(\alpha)ab + c_1(a \otimes b \otimes \alpha)t + c_2(a \otimes b \otimes \alpha)t^2 + \cdots$$

*Then*

(i) *The family $\mathcal{M}$ is associative mod $t^2$ if and only if $c_1$ is a 2-cocycle. That is, $c_1 \in \mathrm{H}^2((A, B, \varepsilon); A)$.*

(ii) *Suppose that $\mathcal{M}$ is associative mod $t^n$. Then $\mathcal{M}$ can be extended to be associative mod $t^{n+1}$ if and only if*

$$c_1 \circ c_{n-1} + c_2 \circ c_{n-2} + \cdots + c_{n-1} \circ c_1 = 0 \in \mathrm{H}^3((A, B, \varepsilon); A).$$

# The Bar Resolution $\mathcal{B}(A)$

**Classical Result**

$$\mathrm{H}^n(A, M) = \mathrm{H}^n(\mathrm{Hom}_{A \otimes A^{op}}(\mathcal{B}(A), M))$$

# The Bar Resolution $\mathcal{B}(A)$

**Classical Result**

$$\mathrm{H}^n(A, M) = \mathrm{H}^n(\mathrm{Hom}_{A\otimes A^{op}}(\mathcal{B}(A), M))$$

**Problem**

Define

$$\mathrm{H}^n((A, B, \varepsilon); M)$$

in an analogous way.

# Simplicial Algebras

**Definition**

A **simplicial $k$-algebra** is a collection of $k$-algebras $A_n$ together with morphisms of $k$-algebras

$$\delta_i^{\mathcal{A}} : A_n \longrightarrow A_{n-1} \quad \text{and} \quad \sigma_i^{\mathcal{A}} : A_n \longrightarrow A_{n+1}$$

such that the simplicial identities are satisfied.

# Simplicial Algebras

**Definition**

A **simplicial $k$-algebra** is a collection of $k$-algebras $A_n$ together with morphisms of $k$-algebras

$$\delta_i^{\mathcal{A}} : A_n \longrightarrow A_{n-1} \quad \text{and} \quad \sigma_i^{\mathcal{A}} : A_n \longrightarrow A_{n+1}$$

such that the simplicial identities are satisfied.

**Example**

Let $A$ be a $k$-algebra, $B$ a commutative $k$-algebra, and $\varepsilon : B \longrightarrow A$ a morphism of $k$-algebras such that $\varepsilon(B) \subseteq Z(A)$. We define the simplicial $k$-algebra $\mathcal{A}(A, B, \varepsilon)$ by setting $A_n = A \otimes B^{\otimes 2n+1} \otimes A^{op}$.

# Simplicial Modules

**Definition**

We say that $\mathcal{M}$ is a **simplicial left module** over a simplicial $k$-algebra $\mathcal{A}$ if $\mathcal{M} = (M_n)_{n\geq 0}$ is a simplicial $k$-vector space together with a left $A_n$-module structure on $M_n$ for all $n \geq 0$ such that we have the following natural compatibility conditions:

$$\delta_i^{\mathcal{M}}(a_n m_n) = \delta_i^{\mathcal{A}}(a_n)\delta_i^{\mathcal{M}}(m_n)$$

and

$$\sigma_i^{\mathcal{M}}(a_n m_n) = \sigma_i^{\mathcal{A}}(a_n)\sigma_i^{\mathcal{M}}(m_n)$$

for all $a_n \in A_n$, for all $m_n \in M_n$, and for $0 \leq i \leq n$.

Let $A$ be a $k$-algebra, $B$ a commutative $k$-algebra, and $\varepsilon : B \longrightarrow A$ a morphism of $k$-algebras such that $\varepsilon(B) \subseteq Z(A)$.

**Example**

We construct the simplicial left module $\mathcal{B}(A, B, \varepsilon)$ over the simplicial $k$-algebra $\mathcal{A}(A, B, \varepsilon)$ bar-like resolution by setting

$$B_n = A^{\otimes n+2} \otimes B^{\otimes \frac{(n+1)(n+2)}{2}}$$

for all $n \geq 0$.
The left module structure, $\delta_i^{\mathcal{B}}$, and $\sigma_i^{\mathcal{B}}$ are all defined in the natural way.

# Co-simplicial Modules

**Definition**

We say that $\mathcal{M}$ is a **co-simplicial left module** over a simplicial $k$-algebra $\mathcal{A}$ if $\mathcal{M} = (M^n)_{n\geq 0}$ is a co-simplicial $k$-vector space together with a left $A_n$-module structure on $M^n$ for all $n \geq 0$ such that we have the following natural compatibility conditions:

$$\delta^i_{\mathcal{M}}(\delta^{\mathcal{A}}_i(a_{n+1}) \cdot m_n) = a_{n+1} \cdot \delta^i_{\mathcal{M}}(m_n)$$

and

$$\sigma^i_{\mathcal{M}}(\sigma^{\mathcal{A}}_i(a_{n-1}) \cdot m_n) = a_{n-1} \cdot \sigma^i_{\mathcal{M}}(m_n)$$

for all $a_{n-1} \in A_{n-1}$, for all $a_{n+1} \in A_{n+1}$, for all $m_n \in M_n$, and for $0 \leq i \leq n$.

# Co-simplicial Modules

## Example

Let $M$ be an $A$-bimodule which is $B$-symmetric. Let $\mathcal{C}(M)$ be the co-simplicial left module over $\mathcal{A}(A, B, \varepsilon)$ obtained by setting $C^n = M$ for all $n$, $\delta^i_{\mathcal{C}} = \mathrm{id}_M$, and $\sigma^i_{\mathcal{C}} = \mathrm{id}_M$. The product is determined by the bimodule structure of $M$ over $A$, which is given by

$$(a \otimes \alpha_1 \otimes \cdots \otimes \alpha_n \otimes \gamma \otimes \beta_1 \otimes \cdots \otimes \beta_n \otimes b) \cdot m_n$$
$$= a m_n b \varepsilon(\alpha_1 \cdots \alpha_n \gamma \beta_1 \cdots \beta_n).$$

# Key Lemma

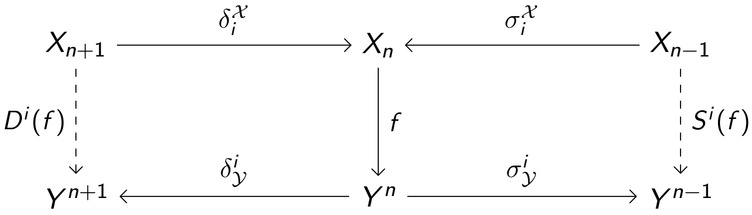
**Lemma (L, Staic, Stancu - 2016)**

*Suppose that* $(\mathcal{X}, \delta_i^{\mathcal{X}}, \sigma_i^{\mathcal{X}})$ *is a simplicial left module over the simplicial k-algebra* $\mathcal{A}$ *and* $(\mathcal{Y}, \delta^i_{\mathcal{Y}}, \sigma^i_{\mathcal{Y}})$ *is a co-simplicial left module over the simplicial k-algebra* $\mathcal{A}$*. Then* $(\mathrm{Hom}_{\mathcal{A}}(\mathcal{X}, \mathcal{Y}), D^i, S^i)$ *is a co-simplicial k-module.*

# Key Lemma

**Lemma (L, Staic, Stancu - 2016)**

*Suppose that $(\mathcal{X}, \delta_i^{\mathcal{X}}, \sigma_i^{\mathcal{X}})$ is a simplicial left module over the simplicial k-algebra $\mathcal{A}$ and $(\mathcal{Y}, \delta^i_{\mathcal{Y}}, \sigma^i_{\mathcal{Y}})$ is a co-simplicial left module over the simplicial k-algebra $\mathcal{A}$. Then $(\mathrm{Hom}_{\mathcal{A}}(\mathcal{X}, \mathcal{Y}), D^i, S^i)$ is a co-simplicial k-module.*



# Main Result

**Theorem (L, Staic, Stancu - 2016)**

$$\mathrm{Hom}_{\mathcal{A}(A,B,\varepsilon)}(\mathcal{B}(A,B,\varepsilon),\mathcal{C}(M))$$

*is a co-simplicial k-module, and*

$$\mathrm{H}^n(\mathrm{Hom}_{\mathcal{A}(A,B,\varepsilon)}(\mathcal{B}(A,B,\varepsilon),\mathcal{C}(M))) = \mathrm{H}^n((A,B,\varepsilon);M).$$

# Main Result

**Theorem (L, Staic, Stancu - 2016)**

$$\mathrm{Hom}_{\mathcal{A}(A,B,\varepsilon)}(\mathcal{B}(A,B,\varepsilon),\mathcal{C}(M))$$

*is a co-simplicial k-module, and*

$$\mathrm{H}^n(\mathrm{Hom}_{\mathcal{A}(A,B,\varepsilon)}(\mathcal{B}(A,B,\varepsilon),\mathcal{C}(M))) = \mathrm{H}^n((A,B,\varepsilon);M).$$

**Remark**

When $B = k$, we get

$$\mathrm{H}^n(\mathrm{Hom}_{A\otimes A^{op}}(\mathcal{B}(A),M)) = \mathrm{H}^n(A,M),$$

the **Hochschild cohomology** of $A$ with coefficients in $M$.

This bar-like resolution $\mathcal{B}(A, B, \varepsilon)$ gives rise to

- a secondary Hochschild homology
- secondary cyclic (co)homologies
- a secondary analogue to Connes' long exact sequence

This bar-like resolution $\mathcal{B}(A, B, \varepsilon)$ gives rise to

- a secondary Hochschild homology
- secondary cyclic (co)homologies
- a secondary analogue to Connes' long exact sequence

# Thank You!